Filter Function Synthesis by Gegenbauer Generating Function

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Abstract: Low-pass all-pole transfer functions with non-monotonic amplitude characteristic in the pass-band and at least $(n-1)$ flatness conditions for $\omega = 0$ are considered in this paper. A new class of filters in explicit form with one free parameter is obtained by applying generating functions of Gegenbauer polynomials. This class of filters has good selectivity and good shape of amplitude characteristics in the pass-band. The amplitude characteristics of these transfer functions have gain in the upper part of pass-band with respect to the gain for $\omega = 0$. This way we have greater margin of attenuation in the upper part of the pass-band. This means a greater tolerance of elements or for elements with given tolerances, greater ambient temperature changes. The appropriate choice of the free parameter enables us to generate filter functions obtained with CHEBYSHEV polynomials of the first and second kind and LEGENDRE polynomials.

Keywords: Gegenbauer polynomials, Filter functions.

1 Introduction

It is well known from literature [1, 2] that the filter function amplitude characteristic can be obtained using classical orthogonal polynomials, such as CHEBYSHEV polynomials of the first and second kind, LEGENDRE polynomials and GEGENBAUER polynomials. They are all orthogonal on the interval $[-1,1]$, with the weighting function of the form $(1-\omega^2)^{\nu-1/2}$, Fig. 1, where ν is a free parameter.

In the complete original papers [3, 4] a new technique for new class of filter function synthesis is described by sets of classical orthogonal polynomials generating functions, Hermite, Legendre, Chebyshev of the first and second kinds, respectively. Further generalization of previous results [3, 4] is described in this paper using powerful Gegenbauer polynomials. Gibbs phenomenon as a consequence of a finite filter function order and classical orthogonal polynomials type and independent kind is also analyzed. Fourier (1768-1830) introduced periodic orthogonal functions, and later set of classical non periodic orthogonal

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polynomials was completed and these are fundamental results of a continental Europe.
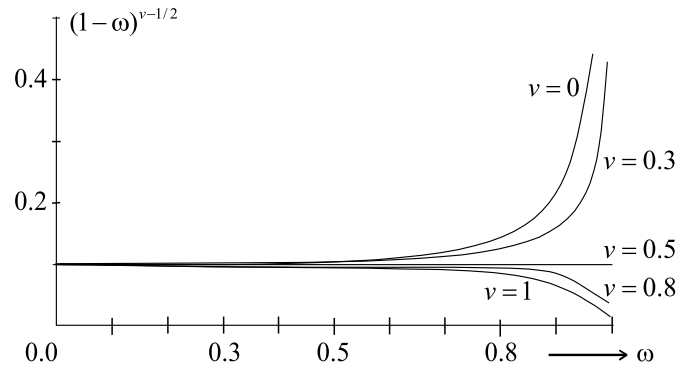

Fig. 1 – *Weighting function of GEGENBAUER polynomial.*

In [5, 6] a useful technique of generating function renormalization that can be used in digital filter function synthesis is given: 1D and 2D (one and two dimension filters).

In the literature [7] Fourier technique is described in an encyclopedic way but there are no examples of extreme problems and attempts for their solving.

In this paper another approach for synthesis of the new class of low-pass selective filters with non-monotonic amplitude characteristic in the pass-band is used. Instead of direct use of polynomials in the synthesis of the characteristic function, the generation of GEGENBAUER polynomials is used. In this way, an explicit expression for the characteristic function is obtained for even or odd order of the filter. The amplitude characteristics of these filters have at least $n-1$ flatness conditions at $\omega = 0$. The group delay is almost constant in the great part of the pass-band. Some particular solutions of this problem are given in the paper [8].

2 Approximation Procedure

Gegenbauer`s polynomials generating function is:

$$g(x,z) = \frac{1}{(1 - 2xz + z^2)^\nu} = \sum_{r=1}^{\infty} C_r^\nu(x) z^r , \qquad (1)$$

where $C_r^\nu(x)$ is Gegenbauer polynomial, z is a complex variable and parameter $\nu \neq 0$.

Gegenbauer polynomial of the rth order is defined [8]:

$$C_r^\nu(x) = \sum_{k=0}^{R} (-1)^k \frac{(\nu)_{n-k}}{k!(n-2k)!}(2x)^{n-k} \ , \qquad (2)$$

where $R = [r/2]$ is the highest integer number not exceeding $r/2$. If the value of the complex variable z is determined so that $g(x,z) = \text{const.}$, then we directly get the expansion of the constant into a series, according to Gegenbauer polynomials:

$$g(x,z)\big|_{z=2x} = \frac{1}{(1-2xz+z^2)^\nu}\bigg|_{z=2x} \equiv 1 \equiv \sum_{r=1}^{\infty} C_r^\nu(x)(2x)^r \ . \qquad (3)$$

Interrupting the expansion of the constant into series according to Gegenbauer polynomials (3), we can easily get the expansion for the square of the amplitude characteristic module

$$\left|A_n^\nu(\omega)\right|^2 = 1 + \varepsilon^2 \frac{\displaystyle\sum_{r=1}^{n} C_r^\nu(\omega)(2\omega)^r}{\displaystyle\sum_{r=1}^{n} C_r^\nu(1)2^r} \ , \qquad (4)$$

where n is the order of the transfer function, and parameter ε^2 defines the attenuation at $\omega = 1$:

$$\left|A_n^\nu(\omega)\right|^2 \bigg|_{\omega=1} = 1 + \varepsilon^2 \ . \qquad (5)$$

According to the known procedure, which consists of the replacement of ω^2 with $-s^2$, and of accepting the zeros of the polynomial $\omega \to \left|A_n^\nu(\omega)\right|^2$ laying in the left half plane of the complex frequency s, because of the stability conditions, transfer function is defined as:

$$F_n(\omega) = \frac{K}{\left(1+\dfrac{s}{\sigma_r}\right)^u \displaystyle\prod_{i=1}^{N}\left(1+\dfrac{s}{\omega_{0i}Q_i}+\dfrac{s^2}{\omega_{0i}^2}\right)} \ , \qquad (6)$$

where: n is the order of the transfer function $N = [n/2]$; K is the constant which determines the attenuation for $\omega = 0$; σ_r is real pole; ω_0 is modulus and Q_i is Q factor of the ith poles pair and $u = n - 2N$.

3   Special Cases

The previous analysis is general, and for some specific values of the parameter ν one can get filter functions, which can be derived using LEGENDRE or CHEBYSHEV polynomials of the first or second kind.

The amplitude characteristics of these filters are as follows:

$$\left|\mathrm{A}_n(\omega)\right|^2 = 1 + \varepsilon^2 \frac{\sum\limits_{r=1}^{n} T_r(\omega)\omega^r}{n} \ , \quad \nu \to 0 \ . \tag{7}$$

$T_r(\omega)$, $r = 1, 2, \ldots, n$ is the CHEBYSHEV polynomial of the first kind [8, 9].

$$\left|\mathrm{A}_n(\omega)\right|^2 = 1 + \varepsilon^2 \frac{\sum\limits_{r=1}^{n} U_r(\omega)(2\omega)^r}{n2^{n+1}} \ , \quad \nu = 0 \ . \tag{8}$$

$U_r(\omega)$, $r = 1, 2, \ldots, n$ is the CHEBYSHEV polynomial of the second kind [9].

$$\left|\mathrm{A}_n(\omega)\right|^2 = 1 + \varepsilon^2 \frac{\sum\limits_{r=1}^{n} P_r(\omega)(2\omega)^r}{2^n - 2} \ , \quad \nu = 0.5 \ . \tag{9}$$

$P_r(\omega)$, $r = 1, 2, \ldots, n$ is the LEGENDRE polynomial [8].

4   Some Characteristics of the Proposed Filter Function

From the expanded version of (3):

$$1 = C_0^\nu(\omega)(2\omega)^0 + C_1^\nu(\omega)(2\omega)^1 + C_2^\nu(\omega)(2\omega)^2 + C_3^\nu(\omega)(2\omega)^3 + \cdots \tag{10}$$

it can be concluded that every term of the sum is an even function of ω, which has r degrees of flatness at the origin $(\omega = 0)$ for the even r , and $r + 1$ degrees of flatness for the odd r . For example:

for $r = 3$, $C_3^\nu(\omega)(2\omega)^3 = a_4\omega^4 + a_6\omega^6$;

for $r = 4$, $C_4^\nu(\omega)(2\omega)^4 = b_4\omega^4 + b_6\omega^6 + b_8\omega^8$;

for $r = 5$, $C_5^\nu(\omega)(2\omega)^5 = c_6\omega^6 + c_8\omega^8 + c_{10}\omega^{10}$; and

for $r = 6$, $C_6^\nu(\omega)(2\omega)^6 = d_6\omega^6 + d_8\omega^8 + d_{10}\omega^{10} + d_{12}\omega^{12}$.

It is well known, that the square amplitude characteristics can be written in the form:

$$\left|\mathrm{A}_n(\omega)\right|^2 = 1 + \varepsilon^2 \frac{\phi_n(\omega^2)}{\phi_n(1)} \ , \tag{11}$$

where $\phi_n(\omega^2) = \sum\limits_{r=1}^{n} C_r^\nu(\omega)(2\omega)^r$ is the characteristic function.

The characteristic function of $(n+1)^{th}$ order has the form:

$$\phi_{n+1}(\omega^2) = \phi_n(\omega^2) + \sum_{i=0}^{n+1} B_i \omega^{2n+2-2i} . \tag{12}$$

Knowing that the least order nonzero term in (12) is:

$$D_0(\omega) = B_0 \omega^{n+1+n+1-2\left[\frac{n+1}{2}\right]} = \begin{cases} B_0 \omega^{n+2} & \text{for } n \text{ even} \\ B_0 \omega^{n+1} & \text{for } n \text{ odd} \end{cases} \tag{13}$$

it can be concluded that the function (12) has all terms of lower degree than $D(\omega)$ equal to zero. In other words, the amplitude characteristic (4) has always n or $(n-1)$ degrees of flatness at $\omega = 0$, irrespective to the value of the parameter ν.

The proposed filter function has one free parameter which controls the selectivity in the transition band, the level of group delay which is constant in the large part of the pass-band and the character of the filter function (amplification or attenuation) in the pass-band.

5   Discussion of the Results

To illustrate the features of the obtained filter functions, the families of ninth order filter functions for $z = 2x$, (3) are shown in **Table 1** for various values of the parameter ν. They have the dominant gain maximum in the pass-band of $0.28\,\text{dB}$ and the loss at the cut-off frequency $a(\omega = 1) = 0.28\,\text{dB}$. The margin of loss in upper part of the pass-band is considerably greater with respect to minimal value, which is on the cut-off frequency.

Table 1
The pole values for $n = 9$.

ν	i	$\sigma_i \pm \omega_i$
-1.6	1	-0.0714843 1.0100426
	2	-0.2715210 0.8173750
	3	-0.4920535 0.5765797
	4	-0.6409021 03003499
	5	-0.6926861
-1.7	1	-0.0739961 1.0177556
	2	-0.2707538 0.8242630
	3	-0.4925202 0.5804565
	4	-0.6425187 0.3022083
	5	-0.6946861
0.5	1	-0.0841322 1.0162744
	2	-0.3045655 0.8365178
	3	-0.5280192 0.5953011
	4	-0.6793970 0.3108953
	5	-0.7322700

Frequency characteristic of the ninth order filters for $v = 0.5$ is shown on Fig. 2 and pass-band characteristic on Fig. 3.
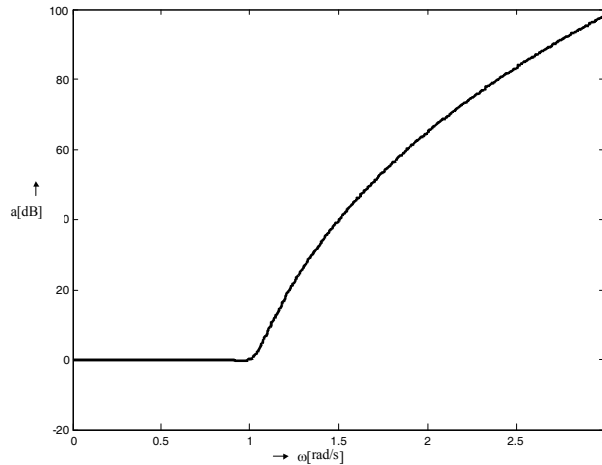

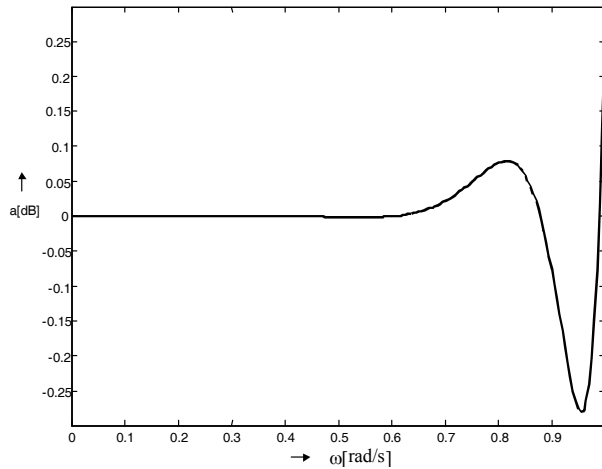**Fig. 2** – *Frequency characteristic of the proposed ninth order filter for $v = 0.5$.*

Fig. 3 – *Pass-band frequency characteristic of the proposed ninth order filter for $v = 0.5$.*

6 Conclusion

In this paper the synthesis of a new class of selective, all-pole low pass even and odd order filters, which have non monotonic amplitude characteristic in the pass-band, is proposed. The procedure is simple and renders an explicit form, which uses GEGENBAUER polynomials. Varying the free parameter v, we can control the value of the maximum amplification in the pass-band or the slope in

the transition band, where the number of flatness of the amplitude characteristic for $\omega = 0$ is at least $(n-1)$ with constant group delay in upper part of the pass-band.

This paper presents general extension of papers [3, 4] by application of powerful Gegenbauer's polynomials. Obtained solutions are unique and have special form of non monotonic amplitude characteristic in the upper part of pass-band. This effect is firstly described in this paper and presents natural occurrence of steep break-off in absolute expansion of classical non periodic orthogonal polynomials generating function. In paper [3, 4] this effect is described in details. As a particular solution for this class of filters we can obtain the filter function with CHEBYSHEV polynomials of the first and second kind, as well as LEGENDRE polynomials.

Comparisons with other type filter functions in time domain and frequency domain are not given in this paper as well as filter realizations with ideal and real components which is correctly done in paper [10, 11]. For a lossy ladder filter the significant problem is to achieve minimal energy losses in the pass-band. Described filter functions may find efficient application in solving such problems.

In papers [12, 13] a new class of low-pass filter functions, which are modification of classical transitional Butterworth-Chebyshev filter with much lower values of the critical pole Q-factor, is shown. Also, in paper [4] the set of numerical examples for filter functions are given which show that values of critical pole Q-factor become lower in wide numerical range which on the other hand present only particular results of this paper. The new class of selective filter functions that is provided may be successfully used for comparisons with transitional filter functions [12, 13].

In further research [14] the part of original technique in Laplace domain for even and odd filter functions synthesis is given. In this paper a direct long consecutive break-off of absolute expansion of constant is used, as well as generating functions of classical orthogonal polynomials.

7   References

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