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The new analytical expression for measurement uncertainty in a measuring of RMS value of AC signals as result of nonideal synchronization

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Abstract: Synchronous sampling allows alternating current (AC) quantities, such as the root mean square (RMS) values of voltage and power, to be determined with very low uncertainties (on the order of a few parts of 10^{-6} [1]). In this a new mathematical expression for estimating measurement uncertainties in nonideal synchronization with fundamental frequency AC signals is presented. The obtained results were compared with those obtained with a high-precision instrument for measuring basic AC values. Computer simulation demonstrating the effectiveness of these new expression, are also presented.

Keywords: Measurement uncertainties, Mathematical expression, Synchronous sampling, AC signals, Nonideal synchronization.

1 Introduction

The synchronous sampling of AC signal enables a highly accurate recalculation of basic electric values in a network. This is possible in the cases when we have a modified signal that is spectrally limited and when we have a sufficient processing time and necessary recalculation capacities.

For this method to be effective, it is necessary to precisely measure the period *T*, as well as to generate the sampling interval $T_S = T/W$, where *T* is the period of the processed signal and *W* is the number of measurements necessary for exact calculation [2, 3]. This method is suitable for sinusoidal and complex-periodical signals with a low harmonic content. There are various sources of error during the synchronous sampling of complex-periodical signals, such as the variable initial time of measurement t_0 [1], the error of the sampling interval generator which depends on the number of samples and the initial phase, the delay of the S/H circuit at a command signal and the effect of the initial phase.

Owing to the issues mentioned above and the nonideal nature of the method, the theoretically obtained discrete sampling moments are not in agreement with

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the experimentally obtained values. Therefore, an additional analysis under the designed conditions and based on the conclusions in [2, 3] should be performed by considering the sensitivity of the procedure suggested in the cases when sampling frequency does not correspond to the actual frequency of fundamental signals. This is performed because synchronous sampling is the method most sensitive to this type of error.

Using a mathematical model for estimating measurement uncertainties we investigate an ideal signal source with a nonideal sampler. The sampler used in this study and in [2, 3] is a high-resolution integrating ADC that operates on the basis of the dual-slope principle. We suppose that the error in estimating the fundamental frequency of the processed signals is Δf . The sampler takes samples synchronously with the same clock reference over an integration time T_i at regular time intervals of length T_a (sampling time). The sampled voltage V_ν (at a time νT_a, where ν is an integer) of the integrating ADC is the mean of the voltage signal $v(t)$ over T_i and is given by [4]:

$$V_\nu = \frac{1}{T_i} \int_{\nu T_a}^{\nu T_a + T_i} v(t)\,\mathrm{d}t\,. \tag{1}$$

The effective value of a signal with a fundamental frequency f (period $T=1/f$) from MN samples (N samples per period over M periods) is given by:

$$\hat{\mathrm{V}}_{RMS} = \frac{1}{\mathrm{Sinc}\left(\dfrac{\pi T_i}{T}\right)} \sqrt{\frac{1}{MN}\sum_{\nu=1}^{MN} \mathrm{V}_\nu^2}\,, \tag{2}$$

where $\mathrm{Sinc}\,x = \sin x / x$ is the function accounting for the transfer function of the sampler in the frequency domain due to (1). Effective AC voltages are estimated using (2) or from the contribution of the spectral line determined by the discrete Fourier transform (DFT) or fast Fourier transform (FFT) of the data from the set of N samples taken over M periods of the sinusoidal voltage generated by the source.

The principle used in [2, 3] is that of calculating the effective voltage and electrical power, defined by relation (2). A practically equivalent method is applied in establishing the active power; its only difference from the above-mentioned method is that the extraction of the square root for obtaining the final value is unnecessary. The zero transition of one of the voltage signals is first detected from the change in the sign of the samples [5]. To avoid false detection due to the noise of high-order harmonic components superimposed on the signal, a minimum delay between two successive transitions based on the expected frequency is assumed. This delay is applied after each valid transition before the acceptance of the successive transition. The time interval between the total numbers of periods of the signal is evaluated. Near the first and last transitions, the samples and their respective times are recorded, and the times of the two transitions are evaluated by the zero intersection of the two segments that best fit these samples. The first frequency is then evaluated as the ratio of the number of periods to the time interval.

The calculation of the uncertainties in nonideal synchronization is exactly performed on the basis of procedure suggested in [2]. The procedure in [6] is widely acknowledged as the best procedure in the calculation of the uncertainties. In some measurement uncertainty calculations, we use algorithms for complex-valued techniques and evaluate them using the simulated data sets [7].

2 Error of Synchronous Sampling in the Case of Nonideal Synchronization

If we only consider the uncertainties Δf resulting from the nonideal synchronization of fundamental signals, and if the subject of processing the voltage signal is of the form:

$$v(t) = V\sin(2\pi f t + \varphi) \tag{3}$$

where f represents the frequency of the basic voltage harmonic and V is its amplitude, $\omega = 2\pi f$ angular frequency and φ is the phase angle. If we perform a calculation to establish the effective (RMS) value using the method in [2, 3] on W equidistant samples from the initial time t_0. then W must satisfy the conditions of synchronous sampling [2, 3 and 8]. The sampling procedure is initiated arbitrarily (Fig.1).

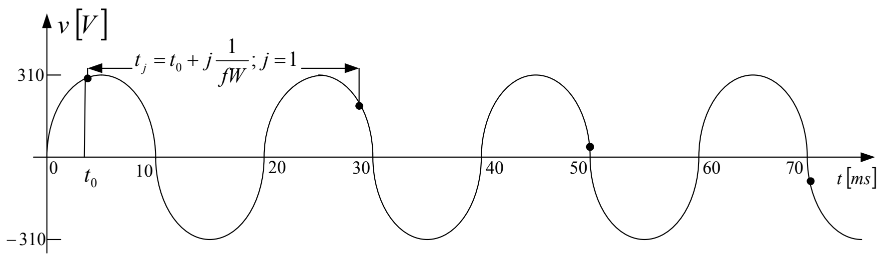

Fig. 1 – *Proposed method of sampling*.

We determine the times at which we take the measurements of the processed value as:

$$t_j = t_0 + j\frac{1}{fW}\,. \tag{4}$$

Here, we introduce the following shortened formula for squaring the signal described in relation (3) and establishing the RMS value of the observed voltage signal:

$$A_1 = \sum_{j=0}^{W-1} \sin^2\left(2\pi f\left(t_0 + \frac{j}{Wf}\right) + \varphi\right), \tag{5}$$

moreover we calculate the value in an actual device, owing to the introduced uncertainties in reading the frequency Δf:

$$A_2 = \sum_{j=0}^{W-1} \sin^2\left(2\pi f\left(t_0 + \frac{j}{W\left(f + \Delta f\right)}\right) + \varphi\right) = \sum_{j=0}^{W-1} \sin^2\left(2\pi f t_0 + \frac{2\pi j\alpha}{W} + \varphi\right), \tag{6}$$

where $\alpha = f/\left(f + \Delta f\right)$ which is the relative deviation from the nominal frequency. In solving the problem related to defining the shape of the signal after the occurrence of the uncertainties in defining the sampling interval, we have to start from the shape defined by relation (6); otherwise, if we suppose that the measured value of the carrying signal (i.e., its frequency) is wrongly read, then we cancel the uncertainties in calculating the basic electrical values (Appendix A). This is reasonable, since the frequency of the processed signal is due to the generator in the observed system.

The uncertainties in calculating the effective value are:

$$\begin{aligned}
\Delta \mathrm{A} = \mathrm{A}_2 - \mathrm{A}_1 &= \frac{1}{2}\sum_{j=0}^{\mathrm{W}-1}\left(\cos\left(4\pi f t_0 + \frac{4\pi j}{W} + 2\varphi\right) - \cos\left(4\pi f t_0 + \frac{4\pi j\alpha}{W} + 2\varphi\right)\right) = \\
&= \frac{1}{2}\sum_{j=0}^{\mathrm{W}-1}\left\{\left(\cos 4\pi f t_0 + 2\varphi\right)\left(\cos\frac{4\pi j}{W} - \cos\frac{4\pi j\alpha}{W}\right) + \right. \\
&\quad \left. + \left(\sin\left(4\pi f t_0 + 2\varphi\right)\right)\left(\sin\frac{4\pi j\alpha}{W} - \sin\frac{4\pi j}{W}\right)\right\} = \\
&= \frac{\cos\left(4\pi f t_0 + 2\varphi\right)}{2}\sum_{j=0}^{\mathrm{W}-1}\left(\cos\frac{4\pi j}{W} - \cos\frac{4\pi j\alpha}{W}\right) + \\
&\quad + \frac{\sin\left(4\pi f t_0 + 2\varphi\right)}{2}\sum_{j=0}^{\mathrm{W}-1}\left(\sin\frac{4\pi j\alpha}{W} - \sin\frac{4\pi j}{W}\right) = \\
&= -\frac{\cos\left(4\pi f t_0 + 2\varphi\right)}{2}\sum_{j=0}^{\mathrm{W}-1}\cos\frac{4\pi j\alpha}{W} + \frac{\sin\left(4\pi f t_0 + 2\varphi\right)}{2}\sum_{j=0}^{\mathrm{W}-1}\sin\frac{4\pi j\alpha}{W},
\end{aligned} \tag{7}$$

where W is the number of measurement of a signal with a known effective value and ΔA is an error in calculus. In the data for A_1, A_2 and ΔA, the amplitude of signal V is intentionally not included; however, this is considered when establishing ultimate uncertainties. We apply the same procedure for establishing the uncertainties in determining the effective value of the signal as the proposed algorithm [2, 3]. In the relation (7) we transform the obtained sums using the *Euler* form of the complex number and by introducing:

$$p_1=\sum_{j=0}^{\mathrm{W}-1}\cos\frac{4\pi\alpha}{W}j;\quad q_1=\sum_{j=0}^{\mathrm{W}-1}\sin\frac{4\pi\alpha}{W}j\quad\Rightarrow$$

$$p_1+iq_1=\sum_{j=0}^{\mathrm{W}-1}\left(e^{\frac{4\pi\alpha i}{\mathrm{W}}}\right)^j=\frac{e^{4\pi\alpha i}-1}{e^{\frac{4\pi\alpha i}{\mathrm{W}}}-1}= \tag{8}$$

$$=\frac{2ie^{2\pi\alpha i}\sin(2\pi\alpha)}{2ie^{\frac{2\pi\alpha i}{\mathrm{W}}}\sin\frac{2\pi\alpha}{W}}=e^{2\pi\alpha i\frac{\mathrm{W}-1}{\mathrm{W}}}\frac{\sin(2\pi\alpha)}{\sin\frac{2\pi\alpha}{W}},$$

where i is the imaginary unit. We express the uncertainties in determining the effective value of the signal as:

$$\Delta\mathrm{A}=-\frac{\cos4\pi ft_0}{2}\frac{\cos\frac{2\pi\alpha(W-1)}{W}\sin(2\pi\alpha)}{\sin\frac{2\pi\alpha}{W}}+\frac{\sin4\pi ft_0}{2}\frac{\sin\frac{2\pi\alpha(W-1)}{W}\sin(2\pi\alpha)}{\sin\frac{2\pi\alpha}{W}}=$$

$$=-\frac{1}{2}\sin(2\pi\alpha)\big(\cos(4\pi ft_0+2\varphi)\sin(2\pi\alpha)+\sin(4\pi ft_0+2\varphi)\cos(2\pi\alpha)\big)= \tag{9}$$

$$=-\frac{1}{2}\sin(2\pi\alpha)\frac{\cos\frac{2\pi\alpha}{W}}{\sin\frac{2\pi\alpha}{W}}\big(\cos(4\pi ft_0+2\varphi)\cos(2\pi\alpha)-\sin(4\pi ft_0+2\varphi)\sin(2\pi\alpha)\big).$$

In the case when phase angle φ=0:

$$\mathrm{A}_1=\sum_{j=0}^{\mathrm{W}-1}\sin^2\left(2\pi ft_0+\frac{2\pi j}{W}\right)=\frac{1}{2}\sum_{j=0}^{\mathrm{W}-1}\left(1-\cos\left(4\pi ft_0+\frac{4\pi j}{W}\right)\right)=\frac{W}{2}$$

$$\Delta\mathrm{A}=-\frac{1}{2}\sin(2\pi\alpha)\left\{\sin(4\pi ft_0+2\pi\alpha)+\cos(4\pi ft_0+2\pi\alpha)\frac{\cos\frac{2\pi\alpha}{\mathrm{W}}}{\sin\frac{2\pi\alpha}{W}}\right\}. \tag{10}$$

If we consider the initial time, at which measuring starts to be t_0=0 (measurements are synchronized with the zero crossings of signals):

$$\Delta\mathrm{A}=-\frac{1}{2}\sin(2\pi\alpha)\left\{\sin(2\pi\alpha)+\cos(2\pi\alpha)\frac{\cos\frac{2\pi\alpha}{W}}{\sin\frac{2\pi\alpha}{W}}\right\}. \tag{11}$$

The absolute obtained relation must satisfy the next inequality:

$$|\Delta\,\mathrm{A}| \le \frac{1}{2}|\sin(2\pi\alpha)|\left(|\sin(2\pi\alpha)| + \cos(2\pi\alpha)\frac{\cos\frac{2\pi\alpha}{W}}{\left|\sin\frac{2\pi\alpha}{W}\right|}\right). \tag{12}$$

Using the conditions

$$f + \Delta f \to f\ (\Delta f \to 0) \quad \text{and} \quad \sin x \approx x\,(x \to 0),$$

we obtain:

$$|\Delta A| \le \frac{W}{4\pi}\cos\frac{\pi}{2W}|\sin(2\pi\alpha)| \tag{13}$$

With the introduction of the amplitude of the processed signal *V,* the definition formula for calculating the effective value of the signal is:

$$U_{RMS} = \sqrt{\frac{1}{W}\sum_{j=0}^{W-1} V^2 \sin^2\left(2\pi f\left(t_0 + \frac{j}{Wf}\right)\right)} \tag{14}$$

and with this equation, we apply:

$$\sqrt{\mathrm{A}_1 + \Delta\,\mathrm{A}} = \sqrt{\mathrm{A}_1}\sqrt{1 + \frac{\Delta\,\mathrm{A}}{\mathrm{A}_1}}; \quad \wedge\,(1+t)^{\frac{1}{2}} = 1 + \frac{t}{2} + \begin{pmatrix}\frac{1}{2}\\2\end{pmatrix}t^2 + \begin{pmatrix}\frac{1}{2}\\3\end{pmatrix}t^3 + \cdots \tag{15}$$

the error in calculating the effective value can thus be presented as:

$$\Delta E = \sqrt{\mathrm{A}_1 + \Delta\,\mathrm{A}} - \sqrt{\mathrm{A}_1} = \sqrt{\mathrm{A}_1}\left(\frac{\Delta\,\mathrm{A}}{2\,\mathrm{A}_1} - \frac{\Delta\,\mathrm{A}^2}{8\,\mathrm{A}_1} + \frac{\Delta\,\mathrm{A}^3}{16\,\mathrm{A}_1} + \cdots\right) \tag{16}$$

By neglecting the higher members in the series, the following error in calculus is obtained:

$$\Delta E \approx \sqrt{\mathrm{A}_1}\,\frac{\Delta\,\mathrm{A}}{2\,\mathrm{A}_1} = \frac{\Delta\,\mathrm{A}}{2\sqrt{\mathrm{A}_1}} \quad \Rightarrow \quad |\Delta E| \le \frac{\Delta\,\mathrm{A}}{2\sqrt{\mathrm{A}_1}}\,. \tag{17}$$

By introducing the amplitude *V* in the relation (17) and by using the definition formula for calculating the effective value, the following is obtained:

$$|E| \le \frac{\mathrm{V}\frac{W}{4\pi}\cos\frac{\pi}{2W}|\sin(2\pi\alpha)|}{2\sqrt{\frac{W}{2}}\sqrt{W}} = \frac{\mathrm{V}}{4\pi\sqrt{2}}\cos\frac{\pi}{2W}|\sin(2\pi\alpha)|, \tag{18}$$

where *E* is the error (absolute) in the calculation of the effective value (it is easily reduced to an error in the calculation of average power) under a supposition that the initial moment of measuring is synchronized with the zero crossing of the signal (t_0=0). The errors in the AC voltage measurement were compared with those given in [9], and were in good agreement.

The following constraints must hold to attain minimum uncertainties:

1) T_a=1/(Wf) must hold at all times for the multiple of two W. This is the condition for the synchronous sampling of the signal with the frequency f generated from a common clock reference.
2) The number of sampled periods M must be an integer multiple of the number of power-line cycles in order to reduce the number of power line interferences. Conditions 1 and 2 prevent artificial spectral components (leakage) from appearing when performing the DFT on the sampled data.
3) The suppression of harmonics of the power line frequency occurs when 1/($T_i f$)>>1 and is an integer.

In the case of complex input signals (with harmonic and nonharmonic components) [8], the uncertainties are evaluated as the superposition's of harmonic errors (with the form defined in relation (18)), and this is expected to be the theme of some future publications.

The total uncertainty of the sampling method is approximately the same as that of the step calibration in the observed frequency range of 46-65 Hz [2]. The presented result (18) enables a more accurate estimation of possible errors in calculating the RMS values of low-frequency AC signals than those presented in [10].

3. Simulation Results

The calculated results were further tested by simulation using the program package Matlab (version 7.0) and module Simulink. The structure of these simulation models is described in detail in [2].

In Fig. 2 a block diagram of the suggested digital measuring system is shown. The system is made of ready-made Simulink models. The unique advantage of using such a program environment or surrounding is that we are able to provide an arbitrary input signal, which is further processed. The signal (comprising two voltage signals or voltage and current signals) is introduced into the circuit for the sample and hold (unit delay), which is located in front of the actual ADC. Then the signal is transferred from the output sample-and-hold circuit into the D flip-flop as a delay element and clocked from the unique signal generator (rectangular series of impulses) for which an arbitrary duty ratio is given. In this manner, the continual signal is measured, and the sample is held constant up to the next measurement or sampling. The next sample is obtained from one of the next periods of the input signal, which is adjusted using the chosen simulation model parameters. Signals are multiplied and then integrated in time, thus obtaining the effective value (or active power). Since it has such an input block, Simulink allows the possibility of introducing a deviation in the frequency of the processed signals.

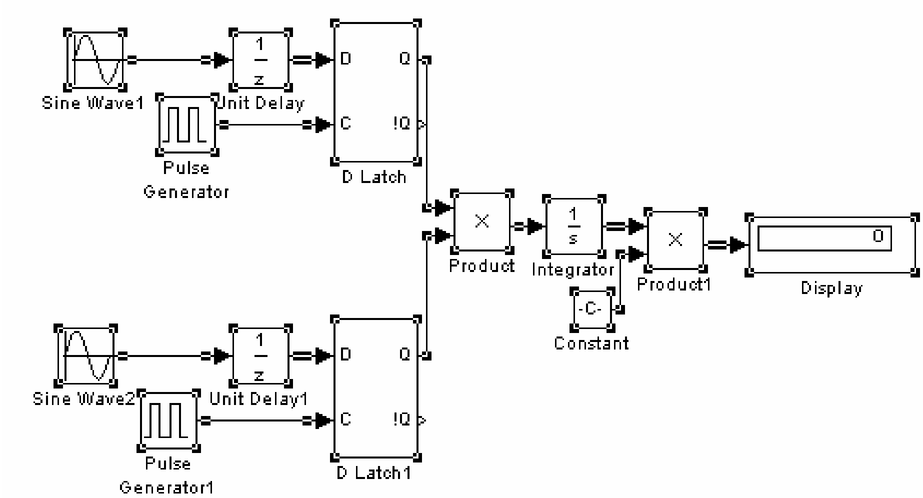

Fig. 2 – *Block diagram of simulation model for measuring effective value* (*or active power*)*, based on measuring concept suggested in* [2].

A separate program has been created in the Matlab. This program enables us (for a known spectral content of the processed voltage and current signals) to establish the desired sampling interval [2, 3, 8]. It also enables us to determine the necessary number of samples to be processed in this manner, so that we can establish the power of the AC signal with a high precision. **Table 1** shows the results obtained by the suggested procedure and the designated program for different cases of nonideal synchronization with a fundamental frequency of the processed signals. The results obtained by applying relation (18) were compared with those obtained using the fabricated instrument described in detail in [2].

From the results given in **Table 1**, it can be concluded that the calculated relation for the uncertainties in the processing of AC signals in the case of nonideal synchronization provides satisfactory results. Thus, we can easily recalculate the uncertainties in the above described case.

The obtained expression for the uncertainties (18) is in agreement with the results and uncertainties in the calculation of the basic AC values in [4, 11, 12, 13, 14].

4 Conclusion

In this study the problem of calculating the uncertainties in nonideal synchronization with a fundamental signal frequency was investigated. An analytical expression was derived, allowing the possibility of establishing in advance the uncertainties in calculating basic AC values, provided that we know the uncertainties in calculating sampling frequency. The obtained results were compared with those obtained using an available, high-precision instrument in licensed laboratories and high-precision sources of voltage and current signals. The findings reveal a very good agreement among the obtained results, with which the correctness of the derived expressions is confirmed. The obtained results were confirmed by simulation.

Table 1

Uncertainties in calculating RMS *value of processing voltage signals obtained using relation* (18)*, and fabricated instrument* [1] ($V = 220\sqrt{2}\cdot[\mathrm{V}]$; f=50Hz; W=40)

Number of measurements	Uncertainties in establishing sampling frequency Δf [Hz]	Uncertainties in establishing effective value of voltage signal, using relation $\lvert E \rvert$ [V]	Uncertainties in establishing effective value of voltage signal, using fabricated instrument $\lvert E^* \rvert$ [V]
1	0.05	0.1097	0.11
2	0.04	0.0878	0.089
3	0.035	0.0768	0.08
4	0.1	0.2193	0.22
5	0.2	0.4378	0.45

5 Reference

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Appendix A

If we consider that the subject of processing is the signal defined by relation (3), and if we perform a calculation to establish the effective RMS value using the definition formula on W equidistant samples from the time t_0, we obtain:

$$\sum_{i=0}^{W-1} \sin^2\left(2\pi t_0 f + \frac{2\pi i}{W}\right) = M_1 \tag{19}$$

where as the value calculated using the fabricated device, due to the introduced error in reading the frequency Δf is:

$$\sum_{i=0}^{W-1} \sin^2\left(2\pi t_0 \left(f + \Delta f\right) + \frac{2\pi i}{W}\right) = M_2 \tag{20}$$

The error in calculating the effective value is determined as:

$$\begin{aligned}\Delta B = M_2 - M_1 = \sin\left(2\pi t_0 \Delta f\right)\Bigg\{ &\sin\left(2\pi t_0\left(2f + \Delta f\right)\right)\sum_{i=0}^{W-1}\cos\frac{4\pi i}{W} + \\ &+\cos\left(2\pi t_0\left(2f + \Delta f\right)\right)\sum_{i=0}^{W-1}\sin\frac{4\pi i}{W}\Bigg\} = 0,\end{aligned} \tag{21}$$

where ΔB is the error in calculation. In the data for M_1, M_2 and ΔB, the amplitude of signal V is intentionally not included. The ultimate expression is equal to zero, because:

$$p = \sum_{j=0}^{W-1} \cos\frac{4\pi j}{W}; \quad q = \sum_{j=0}^{W-1} \sin\frac{4\pi j}{W} \quad \Rightarrow$$
$$p + qi = \sum_{j=0}^{W-1} e^{\left(\frac{4\pi j}{W}\right)i} = \frac{\left(e^{\frac{4\pi i}{W}}\right)^{W} - 1}{e^{\frac{4\pi i}{W}} - 1} = 0. \tag{22}$$