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An Improved Method for Counting 6-Cycles in Low-Density Parity-Check Codes

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Abstract: Since their rediscovery in the early 1990s, low-density parity-check (LDPC) codes have become the most popular error-correcting codes owing to their excellent performance. An LDPC code is a linear block code that has a sparse parity-check matrix. Cycles in this matrix, particularly short cycles, degrade the performance of such a code. Hence, several methods for counting short cycles in LDPC codes have been proposed, such as Fan's method to detect 4-cycles, 6-cycles, 8-cycles, and 10-cycles. Unfortunately, this method fails to count all 6-cycles, i.e., ignores numerous 6-cycles, in some given parity-check matrices. In this paper, an improvement of this algorithm is presented that detects all 6-cycles in LDPC codes, as well as in general bipartite graphs. Simulations confirm that the improved method offers the exact number of 6-cycles, and it succeeds in detecting those ignored by Fan's method.

Keywords: Low-density parity-check (LDPC) code, 6-cycle, Parity-check matrix, Tanner graph.

1 Introduction

Low-density parity-check (LDPC) codes are a class of block codes, they were first proposed by Gallager [1] in the early 1960s. Unfortunately, they were ignored for over three decades due to their decoding complexity that exceeds the capacity of electronic systems at the time, before being rediscovered by MacKay et *al*. [2] in the mid-nineties. Since then, owing to their excellent performance, these codes have become the focus of numerous researchers. Each LDPC code is characterized by a sparse parity-check matrix *H*, i.e., a matrix whose components are elements of GF(2), where the number of ones is low compared to that of zeros. One can represent graphically this matrix by using a graph known as the Tanner graph [3]. This graph is a bipartite graph that consists of bit and check nodes that correspond to the columns and the rows of *H*, respectively. An edge, that connects a bit node to a check node, appears in a Tanner graph if and only if the value of the intersection of the column and the row corresponding to these nodes is equal

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to 1. A cycle in a bipartite graph is a collection of edges which starts with a node and ends to the same node without going through an edge more than once. One can get the length of a cycle by adding up the number of edges it contains, and the shortest length is the girth of the code. Cycles, particularly short cycles, in the Tanner graph is one of the factors that affect the performance of LDPC codes [4 – 6]. Hence, the importance of finding a method to calculate the number of these cycles.

In [7], J. Fan et *al*. have proposed a method to count 4-cycles, 6-cycles, 8-cycles, and 10-cycles in LDPC codes. Unfortunately, this method provides inaccurate numbers of 6-cycles in some LDPC codes. We have noticed this shortcoming after calculating all 6-cycles in several matrices by analyzing the shapes of these cycles (Fig. 1). By comparing the results that we have obtained with those obtained using Fan's method, we have found that Fan's method ignores numerous 6-cycles in two cases, as discussed later in Section 3.

In this paper, we propose an improved version of Fan's method to handle this shortcoming. This improvement allows counting the exact number of 6-cycles in LDPC codes, as well as in general bipartite graphs.

The rest of the paper is organized as follows. In Section 2, we describe Fan's method. The improved method, which is based on Fan's method, is detailed in Section 3, while Section 4 contains the results of computer simulations. Finally, the conclusion is provided in Section 5.

2 Description of Fan's Method

In a given parity-check matrix, a 6-cycle occurs when there exist six 1s in three rows and three columns according to the six shapes shown in Fig. 1.

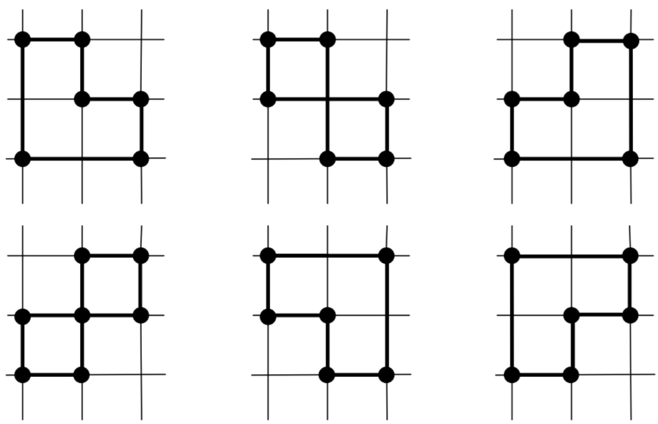

Fig. 1 – *Shapes of 6-cycles in a parity-check matrix* [7].

To calculate the number of 6-cycles in a given $n \times m$ parity-check matrix H, using Fan's method, we need first to calculate the number of three-row combinations which is given by

$$w = \binom{m}{3} = \frac{m!}{3!(m-3)!}. \tag{1}$$

Second, for the i'th combination of three rows, we change the elements equal to 1 in the second and third row, of the matrix H, to 2 and 4 respectively, while those in the first row still unchanged, as shown in Fig. 2.

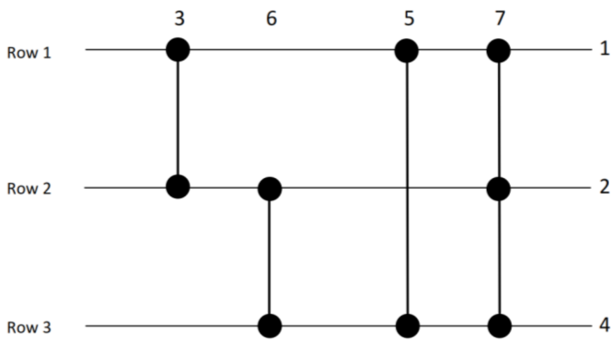

Fig. 2 – *Four kinds of columns in three rows* [7].

Third, by adding up the chosen three rows, we get a sum row containing either the four kinds of columns shown in Fig. 2 or some of them. Let num_3, num_5, num_6, and num_7 denote the number of 3, 5, 6, and 7 in the sum row, respectively. One can calculate the number of 6-cycles in the i'th sum row as follows:

$$\begin{aligned} p_1(i) = {} & num_3 \times num_5 \times num_6 \\ & + num_3 \times num_5 \times num_7 \\ & + num_5 \times num_6 \times num_7. \end{aligned} \tag{2}$$

Finally, the total number of 6-cycles in the matrix H is given by

$$p = \sum_{i=1}^{w} p_1(i). \tag{3}$$

3 Improved Method

In this section, we count all 6-cycles in some parity-check matrices by analyzing the shapes of these cycles (Fig. 1) and compare the obtained results with those obtained using Fan's method. Throughout this section, we will prove that Fan's method fails to detect numerous 6-cycles in two cases.

3.1 First case

This case happens when num_7, in the sum row, is equal to 2. Let

$$H_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 7 & 3 & / & 7 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ \hline / & 5 & 7 & 7 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 7 & / & 7 & 6 \end{pmatrix}.$$

Using Fan's method to calculate the number of 6-cycles in the matrices H_1, H_2, and H_3, we have found that these matrices are 6-cycle free, but indeed, by analyzing the shapes of these cycles given in Fig. 1, each of them contains two 6-cycles.

The 1s that create the first 6-cycle are shown, in bold, in the matrix, H_1, bellow:

$$H_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

For simplicity, we will denote throughout this paper a 6-cycle as

$$C_1 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 4 \end{pmatrix},$$

where the first, second, and third row of C_1 represent the indices of 1s, involved in the first 6-cycle, in the first, second, and third row of H_1, respectively. Thus, the second 6-cycle is

$$C_2 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 4 \end{pmatrix}.$$

Regarding 6-cycles in the second and third matrix, they are given by

$$H_2: \quad C_1 = \begin{pmatrix} 2 & 3 \\ 3 & 4 \\ 2 & 4 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 2 & 4 \\ 3 & 4 \\ 2 & 3 \end{pmatrix},$$

$$H_3: \quad C_1 = \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 3 & 4 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1 & 3 \\ 3 & 4 \\ 1 & 4 \end{pmatrix}.$$

In this first case, the number of 6-cycles is equal to two for each of the combinations (7, 7, 3), (7, 7, 5), and (7, 7, 6). Therefore, the ignored 6-cycles by Fan's method, in the i'th sum row, are

$$\begin{aligned} p_2(i) &= 2\frac{num_7!}{2!(num_7-2)!}\left(num_3+num_5+num_6\right) \\ &= \frac{num_7!}{(num_7-2)!}\left(num_3+num_5+num_6\right). \end{aligned} \tag{4}$$

Thus, the total number of 6-cycles in a given parity-check matrix, in this first case, is

$$p=\sum_{i=1}^{w}\left(p_1(i)+p_2(i)\right). \tag{5}$$

Example 1: *Let H be the parity-check matrix of an LDPC code.*

$$H=\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ \hline 5 & 7 & 3 & / & 7 & 6 & / & 5 & 3 & 5 & / & 6 \end{pmatrix}.$$

Fan's method:

By using (2), *we have* $p_1(1) = 44$. *Thus, the number of 6-cycles is* 44 *cycles.*

Improved method:

By using (4), *we have*

$$p_2(1) = \frac{2!}{(2-2)!}\left(2+3+2\right)=14.$$

The total number of 6-cycles, using (5), *is*

$$p=\sum_{i=1}^{1}\left(p_1(i)+p_2(i)\right)=p_1(1)+p_2(1)=44+14=58.$$

In this example, we notice that Fan's method could not detect 14 *6-cycles, i.e., could not detect p_2 6-cycles.*

3.2 Second case

In this case, num_7 is greater than or equal to 3.

Let

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ \hline 7 & 7 & / & 7 \end{pmatrix}.$$

According to Fan's method, the matrix H has no 6-cycles, whereas it contains six 6-cycles, as shown below.

$$C_1 = \begin{pmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 4 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 4 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 1 & 4 \\ 1 & 2 \\ 2 & 4 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} 1 & 4 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}, \qquad C_5 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 4 \end{pmatrix}, \qquad C_6 = \begin{pmatrix} 2 & 4 \\ 1 & 4 \\ 1 & 2 \end{pmatrix}.$$

We notice that the number of 6-cycles for each combination (7, 7, 7) is equal to six. Thus, the total number of these cycles in the i-th sum row is

$$p_3(i) = 6\frac{num_7!}{3!\left(num_7 - 3\right)!} = \frac{num_7!}{\left(num_7 - 3\right)!}. \tag{6}$$

Furthermore, because this case includes the first case, i.e., num_7 is equal to 2, the total number of 6-cycles is

$$p = \sum_{i=1}^{w}\left(p_1(i) + p_2(i) + p_3(i)\right). \tag{7}$$

Example 2: *Let H be the parity-check matrix of an LDPC code.*

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ \hline 7 & 7 & 3 & 3 & / & 6 & / & 7 & 3 & 5 & / & 6 & 5 & 7 & 7 & / & 7 & 3 & 6 \end{pmatrix}.$$

Fan's method:

The number of 6-cycles is $p_1 = 180$ *cycles.*

Improved method:

By using both (4) *and* (6), *we find that* $p_2(1) = 270$ *and* $p_3(1) = 120$. *Thus, the total number of 6-cycles, using* (7), *is*

$$p = \sum_{i=1}^{1} \left(p_1(i) + p_2(i) + p_3(i) \right) = p_1(1) + p_2(1) + p_3(1) = 180 + 270 + 120 = 570 .$$

In this example, Fan's method has ignored 390 *6-cycles, i.e., has ignored* (p_2 + p_3) *6-cycles.*

The improved algorithm is summarized as follows.

Algorithm: Count the number of 6-cycles

Input: A parity-check matrix *H*.

Output: The number of 6-cycles *p*.

$p = 0$.

for *i* **=** 1 to *w* **do**

$p = p + p_1(i)$.

if (*num₇* = 2**) then**

$p = p + p_2(i)$.

else if (*num₇* >= *3***) then**

$p = p + p_2(i) + p_3(i)$.

end if

end for

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4 Simulation Results

In this section, we compare the improved method, in terms of the number of 6-cycles, with both Fan's and Dehghan's methods [8]. For this comparison, we use the following codes (from [9]).

1. LDPC (TU KL) with N = 96, K = 48 and Rate = 1/2;
2. WiMAX (802.16) with N = 1056, K = 880 and Rate = 5/6;
3. The non-binary LDPC code which is characterized by N = 576, K = 480, Rate = 5/6 and GF(256);
4. The non-binary LDPC code which is characterized by N = 2304, K = 1152, Rate = 1/2 and GF(64).

We denote these four codes by Code A, Codes B, Code C and Code D, respectively. Note that we use the binary images for Codes C and D.

Table 1
Comparing the improved method with that of Fan and Dehghan, in terms of the number of 6-cycles.

Codes	Methods	Number of 6-cycles
Code A	Fan’s Method	216
	Improved Method	216
	Dehghan’s Method	216
Code B	Fan’s Method	16,720
	Improved Method	16,720
	Dehghan’s Method	16,720
Code C	Fan’s Method	8,582,887
	Improved Method	13,891,165
	Dehghan’s Method	13,891,165
Code D	Fan’s Method	1,042,246
	Improved Method	1,764,100
	Dehghan’s Method	1,764,100

Through the results shown in **Table 1**, Fan’s method gives exact numbers of 6-cycles for Codes A and B, which means that these codes do not contain the two cases mentioned in Section 3. Regarding Codes C and D, the results of Fan’s method are different from each of the other two methods. Therefore, the calculation of 6-cycles should include either both cases mentioned in Section 3 or one of them. Furthermore, the number of 6-cycles ignored by Fan’s method is enormous that it reaches 5,308,278 cycles for Code C.

5 Conclusion

An improved version of Fan’s method for counting the number of 6-cycles, in general bipartite graphs, including those corresponding to LDPC codes, has been presented. Simulation results show that the improved method offers the same results as that of Dehghan’s method, which confirm the effectiveness of the solutions detailed in Section 3. Furthermore, the proposed method can be used to find the distribution of 6-cycles in parity-check matrices.

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