SERBIAN JOURNAL OF ELECTRICAL ENGINEERING
Vol. 1, No. 2, June 2004, 187 - 198

Electrostatic Field of Cube Electrodes

Dragutin M. Veličković [1], Alenka Milovanović [2]

Abstract: Electric field and potential distribution in the surroundings of cube electrodes are numerically determined using Equivalent Electrodes Method.

Keywords: Electrostatic field, Potential, Cube electrode, Equivalent Electrodes Method.

1 Introduction

Some time ago the first author suggested a new numerical method, so-called the **Equivalent Electrodes Method** (EEM) [1], for non-dynamic electromagnetic fields and other potential fields of the theoretical physics solving. The first very good results were obtained in [2], when the method was used for calculating of the equivalent radius of uniform antennas. Afterwards, the good results were obtained in the computations of the electrostatic fields [3-6], in the theory of low-frequency grounding systems [7], in the static magnetic field solving [8, 9] and for transmission lines analysis [10-16]. Also, the method was extended to the other potential fields: to heat flow problems [17, 19] and for plan-parallel fluid flow solving [18, 19]. The basic idea of the proposed theory is that an arbitrary shaped electrode can be replaced by a finite system of **Equivalent Electrodes** (EE). Thus it is possible to reduce a large number of complicated problems to the equivalent simple systems. Depending on the problem geometry, the flat or oval strips (for plan-parallel field) and spherical bodies (for three-dimensional fields), or toroidal electrodes (for systems with axial symmetry) can be commonly used. In contrast to the charge simulation method [20], when the fictitious sources are placed inside the electrodes volume, the EE are located on the body surface. The radius of the EE is equal to the equivalent radius of the electrode part, which is substituted. Also the potential and charge of the EE and of the real electrode part are equal. So it is possible, using boundary condition that the electrode is equipotential, to form a system of linear equations with charges of the EE as unknowns. By solving this system, the unknown charges of the EE can be determined and, then, the necessary calculations can be based on the standard procedures. It is convenient to use Green's functions for some electrode, or for stratified medium, in case when the system has several electrodes, or when the multilayer medium exists, and after that the remaining electrodes substitute by EE [21]. In the formal mathematical presentations, the proposed EEM is similar to the moment method form [22], but very important difference is in the physical fundaments and in the process of matrix establishments. So it is very significant to notice that in the application of the

[1] Faculty of Electronic Engineering of Niš, Department of Electromagnetics, Beogradska 14, 18000 Niš, Serbia and Montenegro, Phone: +381 18 529 429, Fax: +381 18 588 399, E-mail: dragan@elfak.ni.ac.yu

[2] Faculty of Technical Engineering, University of Kragujevac, Svetog Save 65, 32 000 Čačak, Serbia and Montenegro, Phone: +381 32 302 739, Fax: +381 32 342 101, E-mail: alenka@tfc.kg.ac.yu

EEM an integration of any kind is not necessary. In the moment method solutions the numerical integration is always present, which produces some problems in the numerical solving of nonelementer integrals having singular subintegral functions. If the charge simulation method is used, the potential can be also put in similar form, but the difference between EEM and charge simulation method is in the choice of the positions of the equivalent charges and in the choice of matching points.

In this paper, the method will be applied for solving electrostatic electrode systems formed by conducting cubes, which are in one and two-layer dielectric, by conducting cube inside spherical cavity and spherical electrode inside cube cavity. Potential Green's function of point charge placed next to the cube electrode will be determined, too.

The obtained numerical results will be compared with existing values realised using other numerical methods (moment method [22, 23], method of sub areas [24] and estimation method [25]) and a very good agreement will be realized. Then it is necessary to notice that EEM gives exact solution in theoretical limit process of infinite EE number. But in real cases, when EE number is finite, existing discontinuities (six peaks and twelve sharp edges) in cube electrode shape produce certain errors, making smaller with EE number increasing.

2 Equivalent Electrode Method Application on Cube Electrodes

In order to the simplicity, the EEM application is presented on the isolated conducting cube of side a, placed in the homogeneous medium of permittivity ε. One of the walls of cube electrode with $4N^2$ squares having sides $A = A/(2N)$, which are formed on the surface, is presented in Fig. 1. These strips can be replaced by small spherical EE of equivalent radius $A_\mathrm{e} = 0.368248A$ [1]. (By using moment method [22], the equivalent radius is $A_\mathrm{e} = 0.373A$.).

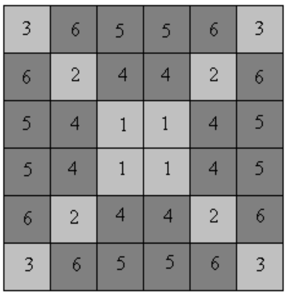

Fig. 1 - *Forming of square strips.*

Due to existing symmetry special care should be taken about diagonal (low darkened) and other (higher darkened) strips in Fig. 1. Therefore, the potential of the cube electrode can be approximately expressed as

$$\varphi = \sum_{n=1}^{M} q_n G_n \,, \tag{1}$$

where $M = N(N+1)/2$ is the number of EE with different charge and G_n is potential Green′s functions of EE charge q_n, which acts as point charge located in the electrical middle point (barycentre) of the strip. If it determines the potential of the diagonal elements, the expression of Green's function has $\alpha_n = 24$ terms having form $1/4\pi\varepsilon R$, where $R = \sqrt{(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2}$ denotes the distance between the barycentre (x_n, y_n, z_n) of the observed strip and the field point (x, y, z). The number of terms in the Green′s function expression in case of no diagonal strips is two times bigger and is $\alpha_n = 48$. N is the number of diagonal EE with different charge on each wall. The number of no diagonal EE is $N(N-1)/2$.

Using boundary condition that the electrode is equipotential, the following system of linear equations is formed,

$$\varphi = U = \sum_{n=1}^{M} q_n G_{nm}, \quad m = 1,2,\ldots,M, \tag{2}$$

where

G_{nm} is finite sum of terms having form $1/4\pi\varepsilon R_{nm}$,

$$R_{nm} = \sqrt{(x_n-x_m)^2 + (y_n-y_m)^2 + (z_n-z_m)^2 + a_{\mathrm{e}n}^2\delta_{nm}},$$

δ_{nm} denotes Kronecker's symbol and $a_{\mathrm{e}n}$ is equivalent radius of $n-$th EE.

After solving this linear equations system, the unknown charges q_n of the EE will be determined. The capacitance of the isolated electrode is

$$C = Q/U = 4\pi\varepsilon a_{\mathrm{e}}, \tag{3}$$

where

$$Q = \sum_{n=1}^{M} \alpha_n q_n \tag{4}$$

is total cube electrode charge and a_{e} is equivalent radius of cube electrode.

The convergence with the number of EE, N, of the presented procedure, when the equivalent radius or capacitance of the isolated cube electrode is determined, is presented in the **Table 1.** These results agree very well with values obtained by using:

a) Moment method [22], $a_{\mathrm{e}} \approx 0.6555a$;

b) Method of sub areas [24], $a_{\mathrm{e}} \approx 0.6611a$; and

c) Estimate method [25], $a_{\mathrm{e}} = a\dfrac{1+\sqrt{3}}{4} \approx 0.6830a$.

Table 1

Equivalent radius of cube electrode for different number of EE, N .

N	a_e/a	N	a_e/a
1	0.661 15	20	0.662 58
2	0.666 71	25	0.662 26
3	0.666 63	30	0.662 04
4	0.666 09	35	0.661 88
5	0.665 55	40	0.661 75
10	0.663 86	45	0.661 65
15	0.663 05	50	0.661 56

3 Examples

In the following text, the method is applied to several typical examples when the cube electrode is placed above conducting plane or it is in two-layer dielectric. Capacitance of the condenser formed by cube electrodes, by cube and spherical electrode and potential Green′s function of point charge next to the cube are also determined.

3.1 Conducting cube above conducting plane

a) Conducting cube of side a is at height h above conducting plane, when the cube base is parallel with plane surface, Fig. 2. The approximate potential can be now presented in form (1), where Green's functions are determined using plan mirror images of point charges above infinite conducting plane.

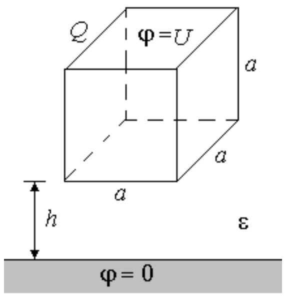

Fig. 2 - *Conducting cube above conducting plane.*

Table 2

Capacitance of cube electrode, $C/4\pi\varepsilon a$ *, for* $h=1.1a$ *and different number of EE,* N .

N	$C/4\pi\varepsilon a$	N	$C/4\pi\varepsilon a$
1	0 .8025	6	0.8439
2	0.8353	7	0.8435
3	0.8426	8	0.8430
4	0.8441	9	0.8426
5	0.8442	10	0.8424

The convergence of capacitance results of conducting cube, placed above conducting plane, when the EE number is different, is presented in the **Table 2**.

The capacitance of the conducting cube placed above conducting plane, when the height, where the cube is placed, is different and the number of EE is determined as $N = 5$, is presented in the **Table 3**.

Table 3

Capacitance of cube electrode, $C/4\pi\varepsilon a$*, when* h/a *is different and* $N = 5$.

h/a	$C/4\pi\varepsilon a$	h/a	$C/4\pi\varepsilon a$
1.05	0.8517	2	0.7697
1.1	0.8442	5	0.7098
1.2	0.8309	10	0.6887
1.3	0.8195	50	0.6712
1.4	0.8096	100	0.6690
1.5	0.8009		

3.2 Conducting cube in two-layer dielectric

b) Conducting cube of side a is in two-layer dielectric, when the cube base is parallel with boundary surface at the height h, Fig. 3. The approximate potential is presented in the form (1), where Green's functions are determined using plan mirror images of point charges in two layers dielectric (Fig. 4),

$$G = \frac{1}{4\pi\varepsilon_1}\left(\frac{1}{r_1} + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\frac{1}{r_2}\right), \text{ in upper layer of permitivity } \varepsilon_1, \quad (5)$$

and

$$G = \frac{1}{2\pi(\varepsilon_1 + \varepsilon_2)}\frac{1}{r_1}, \text{ in lower layer of permitivity } \varepsilon_2. \quad (6)$$

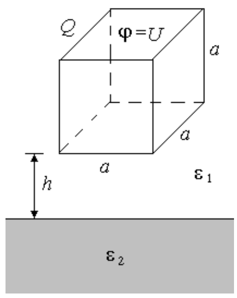

Fig. 3 - *Conducting cube in two-layer dielectric.*

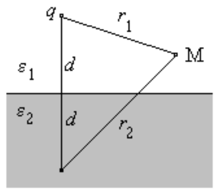

Fig. 4 - *Point charge in two-layer dielectric.*

The capacitance of the cube electrode in two-layer dielectric when $h = 5a$ and different ratio $\varepsilon_2/\varepsilon_1$ is presented in **Table 4**.

Table 4

Capacitance of cube electrode, $C/4\pi\varepsilon_1 a$ *, for* $h = 5a$ *and different ratio* $\varepsilon_2/\varepsilon_1$.

$\varepsilon_2/\varepsilon_1$	$C/4\pi\varepsilon_1 a$	$\varepsilon_2/\varepsilon_1$	$C/4\pi\varepsilon_1 a$
1	0.6668	10	0.7016
2	0.6806	20	0.7055
3	0.6877	50	0.7080
4	0.6920	100	0.7089
5	0.6949		

The capacitances of the cube electrode in two-layer dielectric, when the cube height is different and $\varepsilon_2 = 2\varepsilon_1$ or $\varepsilon_2 = 5\varepsilon_1$, are presented in **Tables 5** and **6**.

Table 5

Capacitance of cube electrode, $C/4\pi\varepsilon_1 a$ *, for* $\varepsilon_2 = 2\varepsilon_1$ *and different ratio* h/a .

h/a	$C/4\pi\varepsilon_1 a$	h/a	$C/4\pi\varepsilon_1 a$
2	0.6979	8	0.6756
3	0.6887	9	0.6747
4	0.6837	10	0.6739
5	0.6806	20	0.6704
6	0.6784	50	0.6683
7	0.6768	100	0 .6675

Table 6

Capacitance of cube electrode, $C/4\pi\varepsilon_1 a$ *, for* $\varepsilon_2 = 5\varepsilon_1$ *and different ratio* h/a .

h/a	$C/4\pi\varepsilon_1 a$	h/a	$C/4\pi\varepsilon_1 a$
1.1	0.7750	2	0.7320
1.2	0.7676	5	0.6949
1.3	0.7610	10	0.6812
1.4	0.7555	50	0.6698
1.5	0.7504	100	0.6683

3.3 Condenser of cube electrodes

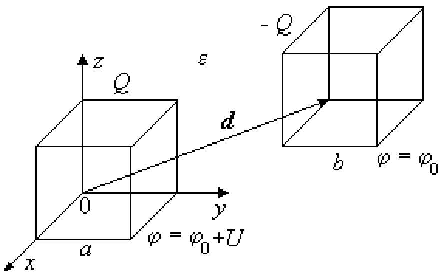

Fig. 5 - *Condenser of cube electrodes.*

The electrodes of cube condenser, having sides a and b, are placed parallel as in Fig. 5. $\boldsymbol{d} = d_x\hat{\boldsymbol{x}} + d_y\hat{\boldsymbol{y}} + d_z\hat{\boldsymbol{z}}$ is radius vector of chosen peak of the right electrode in relation to the corresponding peak of the left cube. The capacitances of parallel cube condensers, for different positions and electrode distances, are presented in Figs. 6 and 7.

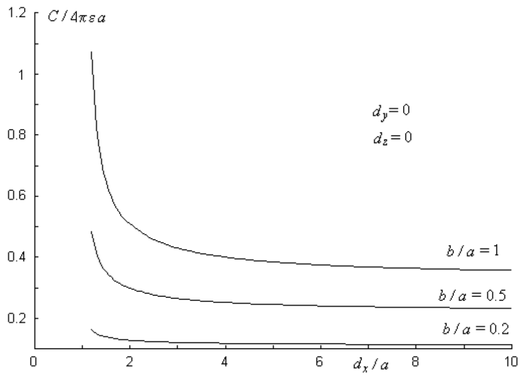

Fig. 6 - *Capacitance of condenser (Fig. 5) versus the distance* d_x *, when* $d_y = d_z = 0$.

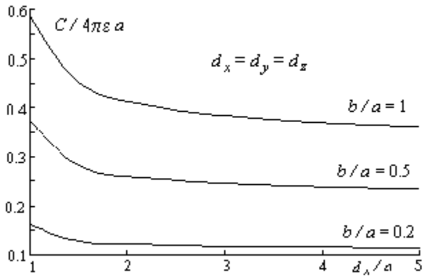

Fig. 7 - *Capacitance of condenser (Fig. 5) versus the distance* $d_x = d_y = d_z$.

3.4 Cube electrode within cube hollow space

Cube electrode within cube hollow space is shown in Fig. 8. In that case the position of the inner electrode peak $0'$ is $x = y = z = (a - b)/2$.

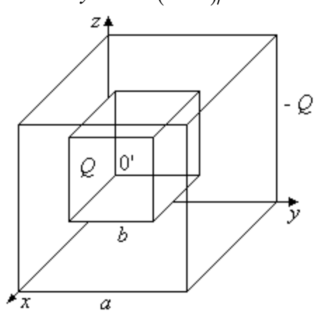

Fig. 8 - *Cube electrode within cube hollow space.*

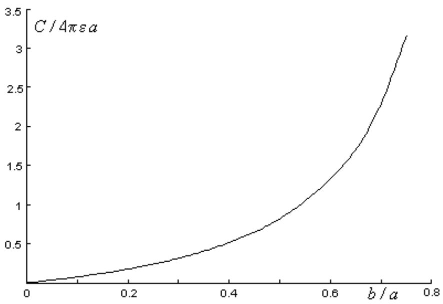

Fig. 9 - *Capacitance of condenser from Fig. 8.*

3.5 Cube electrode inside spherical cavity

Cube electrode of side a is arbitrary placed inside spherical conducting cavity of radius b, Fig. 10. The distance between centres of cube and sphere electrode d is so selected that contact between electrodes is not realized. Now EE are located on cube surface only and the potential can be presented by formula (1), where Green's functions are determined using conducting sphere images of point charges (Fig. 11),

$$G = \frac{1}{4\pi\varepsilon}\left(\frac{1}{r_1} - \frac{b}{h}\frac{1}{r_2}\right). \tag{7}$$

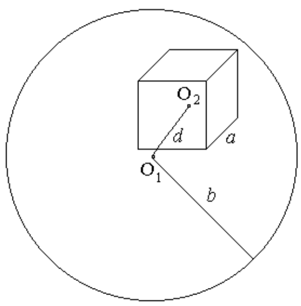

Fig. 10 - *Cube electrode inside spherical cavity.*

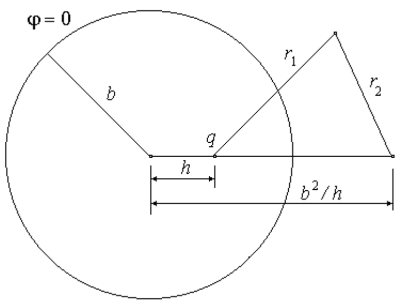

Fig. 11 - *Point charge inside spherical cavity.*

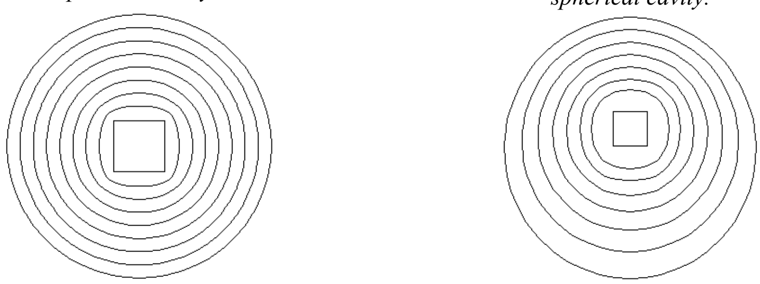

Fig. 12 - *Equipotential line of cube electrode inside spherical cavity.*

Equipotential curves of system from Fig. 10 are presented in Fig. 12. The convergence of the results of capacitance of cube electrode inside spherical cavity with EE number is presented in the **Table 7**, when cube electrode centre coincides with spherical cavity centre and when $a = 0.4b$.

Table 7

Capacitance of system from Fig. 9 for different EE number, N .

EE number, N	$C/4\pi\varepsilon b$
150	0.363 147
600	0.361 884
864	0.361 596
1350	0.361 278
1536	0.361 193
1944	0.361 047

3.6 Spherical electrode inside cube cavity

Spherical electrode of radius b is placed inside cube conducting cavity of side a , as in Fig. 13.

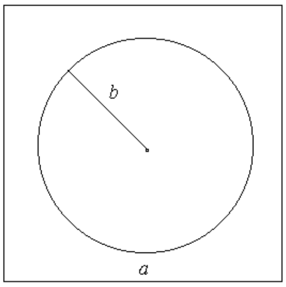

Fig. 13 - *Spherical electrode inside cube cavity.*

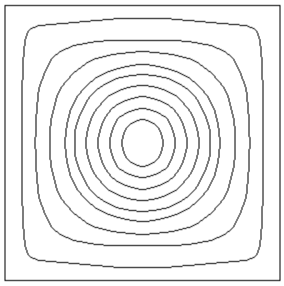

Fig. 14 - *Equipotential lines of spherical electrode in cube cavity.*

The convergence of the results of capacitance of spherical electrode inside cube cavity with EE number is presented in the **Table 8**, when cube electrode centre coincides with spherical cavity centre and when $a = 2.5b$.

Table 8

Capacitance of system from Fig. 13 for different EE number, N .

EE number, N	$C/4\pi\varepsilon b$
150	8.094 084
600	7.982 003
864	7.952 217
1350	7.934 268
1536	7.927 292
1944	7.915 462

3.7 Point charge next to the conducting cube

Point charge q is next to the conducting cube of zero potential as in Fig. 15. Induced charges on the walls of the conducting cube for some position of point charge are presented in **Table 9**.

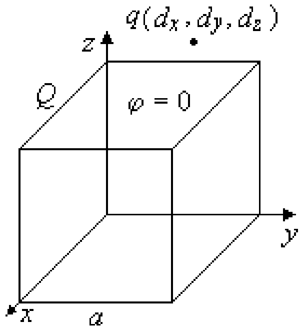

Fig. 15 - *Point charge next to the conducting cube.*

Table 9

Induced charge on the walls of conducting cube for different position of point charge.

Q/q	d_x/a	d_y/a	d_z/a
- 0.861	1	1	1
- 0.386	1.5	1.5	1.5
- 0.642	1.5	0.5	0.5
- 0.591	1.5	0	0.5
- 0.548	1.5	0	0

4 Conclusion

EEM is applied for numerical analysis of electrostatic field of cube electrodes. Conducting cube in one or two-layer dielectric, cube electrode above conducting plane, condenser of two cube electrodes, cube electrode inside spherical cavity, spherical electrode inside cube cavity and point charge next to the cube electrode are observed. Any of these cubes, because of its shape, is rich in sharp edges and peaks, by applying the method gives good convergence depending of the number of used EE. The method is very simple and in the limit process with the number of the EE gives an exact results. It should be pointed out that no numerical integration is necessary and that creating program package for approximate numerical calculations with EEM is very simple. The obtained numerical results are compared with existing values realised using other numerical methods (moment method, method of sub areas and estimation method [25]) and a very good agreement are realized.

5 References

[1] D. M. Veličković: Equivalent Electrodes Method, Scientific Review, 1996 Belgrade, No. 21 - 22, pp. 207 - 248.

[2] D. M. Veličković, Z. Ž. Pantić: A New Numerical Method for Calculating the Equivalent Radius of Uniform Antennas, Sixth Colloquium on Microwave Communication, Budapest, 29th August - 1st September 1978, pp. III.4/25.1 - III.4/25.4.

[3] D. M. Veličković: The Equivalent Electrode Method, 34. Internationales Symposium Teoretische Elektrotechnik, 26 - 31 October 1987, Ilmenau, Band 2, pp. 125 - 128.

[4] D. M. Veličković: General Numerical Program for Plan-parallel Electrostatic Fields Solving, 2th International Conference on Electrostatics ELSTAT 1990, 17 - 22 September 1990, Wroclaw, Poland (MATERIALS SCIENCE, Vol. XVI, No. 4 1990, pp. 89 - 94).

[5] D. M. Veličković: Equivalent Electrodes Method Application for Electrostatic Problems Solving, Third International Symposium on Applied Electrostatics, ПЕС'90, 23rd - 26th October 1990, Niš, Yu, pp. 7 - 27.

[6] D. M. Veličković, Z. Ž. Cvetković: Systems for Generating Homogeneous Electric Field, FACTA UNIVERSITATIS (Niš), Series: Electronics and Energetics, Vol. 14, No. 1, April 2001, pp. 91 - 108.

[7] D. M. Veličković: Equivalent Electrodes Method Application for Grounding Problems Solving, ELEKTROTEHNIKA ELTHB 12 32 (1989) 3 - 4, pp. 149 - 160.

[8] D. M. Veličković, S. R. Aleksić: Magnetic Field Evaluation by Equivalent Electrodes Method, Third International Magnetic Conference IEEE INTERMAG 93', Stockholm, Sweden, 13 - 16 April 1993.

[9] D. M. Veličković, S. R. Aleksić: Magnetic Field Evaluation by Equivalent Electrodes Method, COMPUMAG, Berlin, July 10 - 13, 1995, PF 4 - 7.

[10] D. M. Veličković: General Computer Program for Microstrip Transmission Lines Analysis, Electronic Technology Symposium, 17 - 21 September 1990, Budapest.

[11] D. M. Veličković: General Numerical Program for Line Analysis, Proc. of TELSISK'93, Niš/Yu, 7 - 9 October 1993, pp. 2.25 - 2.32.

[12] D. M. Veličković, Ž. J. Mančić, D. G. Zulkić: Rectangular Coax with Slit and with Rectangular or Circular Centre Conductor, EMC'98 ROMA, Roma, 14 - 18 September 1998, pp. 384 - 389.

[13] D. M. Veličković, Ž. J. Mančić, D. G. Zulkić: Axial Slit on Two Wire Line with Rectangular Shield, Analele Universitatii din Oradea, Fascicola ELECTROTEHNICA, 30 Mai - 1 June, 1998, Baile Felix, Romania, pp. 18 - 23.

[14] D. M. Veličković, S. S. Ilić, D. G. Zulkić: Electromagnetic Field of Coaxial Lines with Axial Slit in a Tunnel, in a Enclosed Bridge or in a Mine Pit, FACTA UNIVERSITATIS (Niš), Series: Electronics and Energetics, Vol. 14, No. 2, August 2001, pp. 167 –185.

[15] D. M. Veličković, S. Ilić: Shielded Strip Line, MMS 2003 Symposium, 6 - 8 May 2003, Cairo.

[16] D. M. Veličković, V. Vučković: General Computer Program for Transmission Lines Design, 48. Internationales Wissenschaftliches Kolloquium, Technische Universität Ilmenau, 22 - 25 September 2003.

[17] D. M. Veličković: The Equivalent Electrodes Method Application in Electroheat, The Third International Conference on Mathematical Modelling in Electroheat, Sarajevo/Yu, October 1991.

[18] D. M. Veličković: General Numerical Program for Plan-parallel Fluid Flow Solving, International Conference on Hydrodynamics of Technological Processes for Materials Production, Sofia, Bulgaria 27 - 31 August 1991.

[19] D. M. Veličković: Equivalent Electrodes Method Application in Fluid and Heat Flow, FACTA UNIVERSITATIS, Series: Mechanics, Automatic Control and Robotics, Vol. 3, No 15, 2003, pp. 951 - 971 (Invited paper).

[20] J. V. Surutka, D. M. Veličković: Some Improvements of the Charge Simulation Method for Computing Electrostatic Fields, Bulletin LXXIV de l'Academie Serbe des Sciences et des Arts, Class des Sciences techn., No. 15, 1981, pp. 27 - 44.

[21] D. M. Veličković: Application of Equivalent Electrodes Method in Multilayer Media, Proc. of 5th International Conference on Applied Electromagnetics, ПЕС 2001, 8 - 10 October 2001, Niš, Yu.

[22] R. F. Harrington: Field Computation by Moment Method, MacMillan, New York, 1968.

[23] Kinseth Reitan, Thomass James Higgins: Calculation of the Electrical Capacitance of a Cube, Journal of Applied Physics, Vol. 22, No. 2, February 1951, pp. 223 - 226.

[24] Er-Wai Bay, Karl E. Lonngren: On the capacitance of a cube, Computers and Electrical Engineering, 28 (2002), pp. 317 - 321.

[25] D. M. Veličković, A. Veličković: Approximate solution of electrostatic problems, STATICKA ELECTRINA, Krakov, October 1980, pp. 7 - 16.