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Generalization of Helmholtz Coil Problem

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Abstract: The primary intent of this work is to propose a simple analytical method for designing coil systems for homogeneous and gradient magnetostatic field generation. Coil system consists of two identical coaxial (regular) polygonal current loops. In the space between the loops, there is nearly homogeneous or nearly linear distribution of the magnetic field along the axes depending on the currents' direction. First, we derived a suitable, simple and general expression for the magnetic field along the axes due to a polygonal current loop. We emphasize the importance of the role of this expression for further analysis. The total on-axes magnetic field is the result of superposition of the magnetic fields that each loop generates separately. The proper distance between the loops and the current orientation make the system to become either Helmholtz coil or anti-Helmholtz coil. In this paper we give exact, analytical and general expression for this optimal distance that provides the magnetic field to be homogeneous (linear) as much as possible. We based our study on Taylor series expansion of the total magnetic field, demanding that the first contaminating term must be canceled, in both, symmetric and asymmetric case.

Keywords: Analytical models, Magnetic fields, Taylor series, Polygonal coils.

1 Introduction

The Helmholtz coil is the simplest current configuration for producing a relatively constant magnetic field. This terminology is usually reserved for two current coaxial circular loops with the same radius that is equal to the distance between the coil plains. Currents flowing in opposite directions, in the same arrangement, produce gradient magnetic field. When the loops are spaced by a proper (optimal) distance, the configuration is known as anti-Helmholtz or (Maxwell) gradient coil. Despite the simplicity, the interest in these problems lasts century and a half, primarily due to the necessity of cancellation of external magnetic fields in a variety of applications. The homogeneous fields, as well as the gradient fields, are widely used in measurements, biomedical research, calibration of probes and sensors, etc.

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Spreading the region of homogeneity and reducing variation of the field on the axis are the tasks of general interest. Different approaches are used to achieve these goals, including increasing the number of loops, loops with different radii or both. Certain studies are related to a different geometry of loops [1], and the square shaped loop is mostly under consideration [2 – 5].

Typically, excluding a few examples, the magnetic field and the optimal distance between the loops are determined using ready-made software packages [2, 4, 6]. Also, some results are based on experimental research [7]. In this paper, we present a purely analytical procedure for obtaining polynomial equations that describe the requirements for homogeneity as well as for linearity. The real positive root of the corresponding equation is the optimal distances in the case of either the Helmholtz or anti-Helmholtz polygonal coil. Clearly, the procedure is analytical up to the level at which the final polynomial equation must be solved numerically.

2 Polygonal Current Loop

We begin our study with an expression for the magnetic field of a thin straight current carrying wire of finite length in terms of initial and final angles,

$$B = \frac{\mu_0 I}{4\pi r}\left(\cos\theta_1 - \cos\theta_2\right), \tag{1}$$

where r is the perpendicular distance of the field point O_1 to the wire, Fig. 1. This expression follows from the basic form of Biot-Savart law for a current filament.

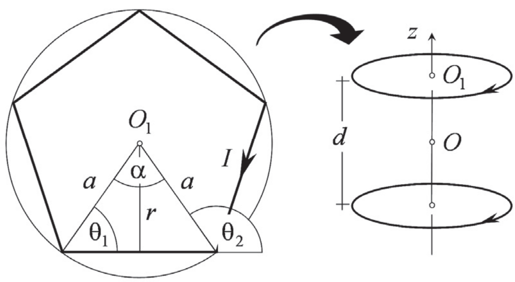

Fig. 1 – *Current filament, current loop and pair of current loops*

In special case, in which the field point is at the axis of symmetry of the conductor, formula for magnetic field can be rewritten in a more illuminating form:

$$B = \frac{\mu_0 I}{2\pi a}\tan\frac{\alpha}{2}, \tag{2}$$

where a is distance between the field point and any of the end points and α is the angle between the lines drawn from the field point to the ends of the wire.

The derived formula can be used for determination of the magnetic field at the center of a regular polygonal current loop. This can be done multiplying (2) by n, where n is the number of vertices. The angle α now becomes central angle of an $n-$gon, $\alpha = 2\pi / n$ and a is a radius of a circumscribed circle,

$$B_{n0} = \frac{\mu_0 I}{2a} \frac{\tan(\pi / n)}{\pi / n}. \tag{3}$$

Obviously in the limiting case $n \to \infty$, n-gon tends to circle and (3) leads to the well known result for magnetic field in the centre of the circle,

$$B_{C0} = \lim_{n\to\infty} B_{n0} = \frac{\mu_0 I}{2a}. \tag{4}$$

Obtaining formula for the magnetic field on the axis of the polygonal current loop is a bit of a challenge because it will be necessary to have a suitable form of expression for further calculations. Now we will consider the magnetic field at the arbitrary point that belongs to *z*–axis. Due to axial symmetry radial components of magnetic field cancel each other out. After some vector calculus, formula (3) becomes expression for the total magnetic field which is in *z*–direction,

$$B_n(z) = B_{C0} \frac{\sin(2\pi / n)}{2\pi / n} f_n(z), \tag{5}$$

where

$$f_n(z) = \frac{a^3}{(z^2 + a^2 \cos^2(\pi / n))\sqrt{z^2 + a^2}}. \tag{6}$$

A similar, but no so refined, formula can be found in [8]. As before, verification can be done by finding a limiting value, i.e. $n \to \infty$. It is clear that the above formula degenerates to a known result for magnetic field of circular loop. The second verification can be done for square shaped current loop. However, there is a slight snag here. Namely, all known results involve side length of the square so variables have to be adjusted to obtain agreement. Only for this purpose a should be substituted by $a/\sqrt{2}$. After that, results will be matched. Here and further on, we will use the radius of the circumscribed circle a.

It is convenient to introduce dimensionless quantities. First, we will normalize the distance $z \to z / a$, which is the same as taking $a = 1$. It should be noted that we use the same notation z for the dimensionless distance quantity and its dimensional counterpart. In that manner, (6) becomes

$$f_n(z) = \frac{1}{(z^2 + c^2)\sqrt{z^2 + 1}}\,,\ c = \cos(\pi / n)\,. \tag{7}$$

3 Helmholtz Pair

We will consider two equal regular polygonal loops separated by a distance d between the centers, carrying the same amount of current in the same direction. The loops are placed symmetrically, so $z = 0$ is the midpoint of the loops. In spite of polygonal shape of the loops, the term Helmholtz pair still remains valid. More precisely, the distance d should be determined to provide the magnetic field, in the region between the loops, to be homogeneous as much as possible. This optimal distance makes the loops to become a Helmholtz pair.

The magnetic field at any point along the axis of the loops can be calculated by summing the contribution from each of the separate loop. The magnetic field originating from each of the loops in particular is proportional to the value of the function (7) at the point in which the field would be calculated. In dimensionless notation, this means that argument z is shifted for $\pm\varsigma$, where $\varsigma = d / (2a)$ is the ratio of half the distance between the two loops and the radius of the circumscribed circle. Finally, the magnetic field along the axis of the loops is

$$B_n(z) = B_{C0}\frac{\sin(2\pi / n)}{2\pi / n}F_n(z,\varsigma)\,, \quad \varsigma = \frac{d}{2a}\,, \tag{8}$$

$$F_n(z,\varsigma) = f_n\left(z - \varsigma\right) + f_n\left(z + \varsigma\right). \tag{9}$$

Further studies are based on the expansion of the function (9) into Taylor series about $z = 0$. The magnetic field is approximately equal to zero term in the Taylor expansion and it is given by

$$B_n(0) = B_{C0}\frac{\sin(2\pi / n)}{2\pi / n}\frac{2}{(\varsigma^2 + c^2)\sqrt{\varsigma^2 + 1}}\,. \tag{10}$$

This term is a constant, for given polygon and particular value of ς, and has the dominant influence on the magnetic field strength. Due to the symmetry of the problem, the first and all odd derivatives are equal to zero at $z = 0$. The first term that contributes to inhomogeneity of the magnetic field is the second order derivative in Taylor series expansion. In order to achieve the best homogeneity, at this level of problem solving, we have to cancel this term. The leading equation for determining the optimal distance between loops arises from the requirement that the second derivative in Taylor series expansion is equal to zero at the point $z = 0$.

$$\begin{aligned}12\varsigma^6 &+\\ (15+6c^2)\varsigma^4 &+\\ (6-2c^2+2c^4)\varsigma^2 &-\\ (2c^2+c^4)\varsigma^0 &= \quad 0.\end{aligned} \tag{11}$$

The real positive root ς_{opt} of the equation (11) gives the required optimal distance. At this distance, the loops become the Helmholtz pair and the produced magnetic field is constant up to the third power of z.

4 Anti-Helmholtz Pair

An anti-Helmholtz coil is constructed like a Helmholtz coil, except the current in the two loops flows in opposite directions. The magnetic field along the axis of symmetry is described by (8) with the sign reversed in the second term of (9). Thus, we have

$$F_n(z,\varsigma) = f_n(z-\varsigma) - f_n(z+\varsigma). \tag{12}$$

At the point $z=0$ the magnetic field is equal to zero, $F_N(0,\varsigma)=0$. Due to the asymmetric nature of the coil, all even derivatives vanish at the same point from Taylor expansion of (12). In the vicinity of this point magnetic field varies linearly as a function of a position. This (spatial) variation of the field is called a field gradient,

$$B_n^{(1)}(z) = B_{C0}\frac{\sin(2\pi/n)}{2\pi/n}\frac{2\varsigma(3\varsigma^2+c^2+2)}{(\varsigma^2+c^2)^2(\varsigma^2+1)^{3/2}}\frac{z}{a}. \tag{13}$$

Therefore, anti-Helmholtz coil is a convenient structure for generating field gradients. A high degree of linearity is desirable. The term of the lowest order, which disturbs the linearity is the third derivative. We demand that this be equal to zero, which is consistent with the procedure applied to the Helmholtz coil. From there we get a leading polynomial equation,

$$\begin{aligned}20\varsigma^8 &+\\ (35+10c^2)\varsigma^6 &+\\ (28-21c^2+8c^4)\varsigma^4 &+\\ (8-24c^2-11c^4+2c^6)\varsigma^2 &-\\ (8c^2+4c^4+3c^6)\varsigma^0 &= \quad 0.\end{aligned} \tag{14}$$

The real positive root ς_{opt} of the equation (14) gives the required optimal distance. At this distance, the structure becomes the anti-Helmholtz pair. Thus, a magnetic field between the loops is linear up to the fourth power of z.

5 Numerical Results

The leading equations (11) and (14) for optimal distances between the loops, in symmetric and asymmetric cases, are polynomials of the degree six and eight, respectively. Using a quadratic substitution, equations can be solved as equations of the degree three and four, respectively. Each of the above equations has a real and positive root for any $0.5 \le c \le 1$, which we are looking for in order to determine the required optimal distance ς_{opt}. It is convenient to choose, and we did so, the method of bisection, since the solution is bracketed between zero and limiting value obtained for a circle. The optimal distance for variety of polygons we present in **Table 1**, for the Helmholtz and anti-Helmholtz coil as well. It is obvious that there is no need to examine polygons for $n > 15$. These results would be very, very close to the well-known result for a circle. The graphs of the ratio $B_n(z)/B_{C0}$ as a function of the normalized distance z/a for various numbers of vertices are shown in Fig. 2 and Fig. 3 for the Helmholtz and anti-Helmholtz configuration, respectively.

Table 1
The Optimal Normalized Distances.

n	**Polygon**	$\varsigma_{opt} = d/(2a)$	
		Helmholtz, (11)	**Anti-Helmholtz, (14)**
3	Trigon	0.28300	0.49243
4	Tetragon	0.38502	0.66881
5	Pentagon	0.42904	0.74416
6	Hexagon	0.45178	0.80111
8	Octagon	0.47349	0.82358
12	Dodecagon	0.48842	0.84600
15	Pentadecagon	0.49263	0.85326
$\to \infty$	Circle	0.50000	0.86603

First, let's examine more closely polynomial (11) that leads to homogeneous magnetic field. In the simplest case $c = 1$, the case of a circle, this equation has the exact analytical solution, $\varsigma_{opt} = 1/2$. This is the widely known condition of the Helmholtz coil, $d = a$. In all the other cases leading equation has to be solved numerically. The second check is referring to, not less known, the optimal distance between two square loops, $d = 0.5445s$, where s is a side of a square loop.

This result is already known for some time, but it seems that [3] is the most referenced article. In particular, here we get $\varsigma_{opt} = 0.38502$, which is the same as previously indicated result, bearing in mind that we are working with a radius of the circumscribed circle and $s = a\sqrt{2}$. Anyway, both results shown here serve to confirm the accuracy of the proposed method.

Now, we will consider an anti-Helmholtz configuration. Equation (14) has the exact analytical solution in the case of a circle, $\varsigma_{opt} = \sqrt{3}/2$. As far as we are informed, this is the only known result referring to a simple gradient coil.

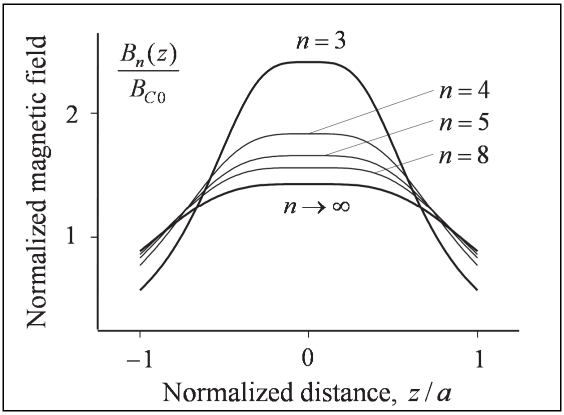

Fig. 2 – *On-axes normalized magnetic field for polygonal Helmholtz configuration.*

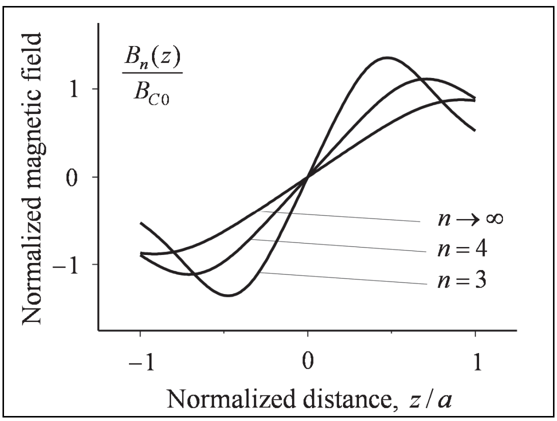

Fig. 3 – *On-axes normalized magnetic field for polygonal anti-Helmholtz configuration.*

The homogeneity of the magnetic field δ inside the Helmholtz configuration is a measure of the variability of the magnetic field within a region between the loops. In the general case the generated magnetic field has the axial and the radial component, B_z and B_r respectively. Ideally, in a three-dimensional region of interest, the magnetic field vector should have a constant axial and zero radial component. We can express the magnetic field components as the required field $B(0)$ and error terms in the form [10]

$$B_z = B(0) + \delta_z B(0)\,, \quad B_r = \delta_r B(0)\,, \tag{15}$$

where $\delta_z < 1$ and $\delta_r < 1$ are small positive constants. The deviation from the homogeneity of the magnetic field is

$$\delta = \frac{|\,B(0) - \sqrt{B_z^2 + B_r^2}\,|}{|\,B(0)\,|} = |\,1 - \sqrt{(1+\delta_z)^2 + \delta_r^2}\,| \approx \delta_z + \frac{1}{2}\delta_r^2\,. \tag{16}$$

In the case of nearly homogenous field, the second term has negligible influence on the total deviation and that the radial component of the magnetic field can be neglected. In accordance with the performed calculations, the relative deviation from homogeneity is defined only for points on the axis of the coil [2, 9],

$$\delta(z) = \left| 1 - \frac{B_n(z)}{B_n(0)} \right| 100\,[\%]\,. \tag{17}$$

Here, $B_n(z)$ and $B_n(0)$ should be calculated from (8), (9) and (10).

Due to the absence of adequate data in the literature, the only verifications that we were able to implement are those for circular and square geometry of the coil. Relative deviation from homogeneity for various polygonal shapes is given in **Table 2**. More precisely, we give the ratio $|\,z/a\,|$ for selected values of homogeneity. Because the square is mostly the only polygon considered in literature, the authors always deal with a side length. As before, certain data had to be recalculated and we did so for the column labeled with 4** in **Table 2**. The last row in this table (denoted as Level) contains values of normalized magnetic field obtained via (10). In other words, this constant multiplied by B_{C0} is the on-axes magnetic field in the given scope of $|\,z/a\,|$ with the defined homogeneity δ.

It is worthwhile to point out that in [2] the homogeneity of magnetic field is approximately calculated. Regardless, the results show a surprisingly good agreement. Permitted variations in the homogeneity of the magnetic field depend on the application of the coil. For most applications, variation is within a small, well known range, as it is shown in **Table 2**.

A similar analysis can also be performed in the asymmetric case, this time for the range of linearity. At the midpoint of the system, magnetic field is linear function of distance (13). The desirable characteristics of the magnetic field are that it has a constant slope or gradient to the maximum extent. The relative deviation from linearity is defined then by

$$\delta(z) = \left| 1 - \frac{B_n(z)}{B_n^{(1)}(0)} \right| 100\,[\%], \tag{18}$$

where, $B_n(z)$ and $B_n^{(1)}(z)$ should be calculated from (8), (12) and (13). The results of the calculation are given in **Table 3**. In the last row (denoted as Slope) we present the values of the slopes that are encapsulated in (13). In fact, the deviation from linearity is the deviation from the value that is obtained by applying (13). Unfortunately, we did not find the appropriate data for comparison.

Table 2

The Normalized Field Homogeneity for Different n.

$\|z/a\|$ for specified scope of homogeneity δ							
$\delta[\%]<$	**3**	**4**	**4****	**6**	**8**	$\to\infty$	$\to\infty$ *
0.01	0.056	0.075	0.075	0.088	0.092	0.097	0.097
0.05	0.084	0.113	0.112	0.132	0.138	0.145	0.145
0.10	0.100	0.134	0.134	0.157	0.164	0.174	0.173
0.50	0.150	0.203	0.202	0.237	0.248	0.262	0.262
1.00	0.180	0.243	0.243	0.284	0.298	0.314	0.314
5.00	0.280	0.378	0.378	0.442	0.463	0.489	0.488
Level	2.411	1.833		1.580	1.510	1.431	

* Results from [2]

** Results from [2] multiplied by $\sqrt{2}/2$

Table 3

The Normalized Field Linearity for Different n.

$\|z/a\|$ for specified scope of linearity δ					
$\delta[\%]<$	**3**	**4**	**6**	**8**	$\to\infty$
0.01	0.069	0.093	0.048	0.140	0.119
0.05	0.103	0.138	0.123	0.186	0.177
0.10	0.123	0.165	0.280	0.215	0.211
0.50	0.184	0.246	0.346	0.310	0.316
1.00	0.218	0.293	0.381	0.365	0.376
5.00	0.327	0.439	0.548	0.542	0.565
Slope	3.610	2.094	1.520	1.421	1.283

6 Conclusion

The major focus of this paper was our endeavor to find a general solution of the Helmholtz and anti-Helholtz coil problem. The general expression for determining the optimal distance between the polygonal current loops is shown here. The proposed solution is validated by comparing the calculated results with a small number of known data where we found a perfect matching

This paper is a cornerstone for a variety of unsolved problems. Almost without changes, our results can be applied to coil with two different polygonal loops, e.g. a circle and a square loop. With a little effort and negligible corrections the method can be applied to systems with three or more loops.

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