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Sensitivity Analysis of Electromagnetic Quantities in Time Domain by Means of Fem

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Abstract: Sensitivity analysis belongs to the most important tools in optimization theory. It determines the dependence of global or local electromagnetic quantities on geometrical and physical parameters expressed in the form of an objective function. For several objective functions the sensitivity may be directly calculated differentiating the objective function versus one of material or geometric parameters. Such approach needs large computational effort, especially while evaluating in time domain.

This paper presents effective methods for computing of sensitivity of nodal potentials in finite elements versus perturbations in conductivity of analyzed model in time domain. Derived equations are based on the method of stiffness and mass matrices derivative and Tellegen's theorem known from circuit theory and have been expanded on field theory.

Numerical example presented in the paper shows sensitivity of voltage induced in measurement coil versus variation of electrical conductivity in single finite element as function of time. The proposed methods calculate the sensitivity versus all finite elements in area of analysis at once. On the basis of sensitivity information the iterative algorithm for identification of shape and conductivity distribution of material flaws could be applied.

1 Introduction and definitions

When analyzing electromagnetic fields using finite element method in time domain the following equations system is obtained

$$[\mathbf{H}]\cdot[\mathbf{A}(t)]+[\mathbf{C}]\cdot\frac{\partial}{\partial t}[\mathbf{A}(t)]=[\mathbf{I}(t)], \tag{1}$$

with: [**H**] - stiffness matrix and [**C**] - mass matrix of finite elements. All material and geometrical properties of the model are contained in matrices [**H**] and [**C**], vector [**I**(t)] includes excitation nodal currents and [**A**(t)] is nodal magnetic potential vector. Applying time stepping method results in

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$$\left([\mathbf{H}]+\frac{1}{\Delta t}\cdot[\mathbf{C}]\right)\cdot[\mathbf{A}(t_i)]=[\mathbf{I}(t_i)]+\frac{1}{\Delta t}\cdot[\mathbf{C}]\cdot[\mathbf{A}(t_{i-1})], \quad i=1..n\,. \tag{2}$$

$[\mathbf{A}(t_0)]$ is initial condition vector. In most cases of field penetration into conducting region, this vector may be set to zero. Next vectors of results are calculated in iterative process of time stepping and buffered in computer's memory.

The simplest method of sensitivity calculation is differential method. It depends on calculating potential change $\Delta\mathbf{A}(t_i)$ with relation to conductivity's variation $\Delta\gamma$ in defined area

$$[\mathbf{S}(t_i)]=\frac{[\mathbf{A_0}(t_i)]-[\mathbf{A_1}(t_i)]}{\Delta\gamma}=\frac{[\Delta\mathbf{A}(t_i)]}{\Delta\gamma}. \tag{3}$$

After analysis of the original model this method demands independent analysis with FEM in time domain for every considered element, it means, for the model regarding conductivity variation inside the element. For this reason such approach seams to be not efficient.

2 Method of stiffness and mass matrices derivative

Variation of electrical conductivity of the material causes changes of magnetic vector potential $[\mathbf{A}(t_i)]$ as follows

$$\frac{\partial}{\partial\gamma}\left(\left([\mathbf{H}]+\frac{1}{\Delta t}\cdot[\mathbf{C}]\right)\cdot[\mathbf{A}(t_i)]\right)=\frac{\partial}{\partial\gamma}\left([\mathbf{I}(t_i)]+\frac{1}{\Delta t}\cdot[\mathbf{C}]\cdot[\mathbf{A}(t_{i-1})]\right). \tag{4}$$

Neglecting high order small terms, the following equation could be obtained

$$\frac{\partial}{\partial\gamma}\left([\mathbf{H}]+\frac{1}{\Delta t}\cdot[\mathbf{C}]\right)\cdot[\mathbf{A}(t_i)]+\left([\mathbf{H}]+\frac{1}{\Delta t}\cdot[\mathbf{C}]\right)\cdot\frac{\partial}{\partial\gamma}[\mathbf{A}(t_i)]=$$
$$=\frac{\partial}{\partial\gamma}[\mathbf{I}(t_i)]+\frac{1}{\Delta t}\cdot\frac{\partial}{\partial\gamma}[\mathbf{C}]\cdot[\mathbf{A}(t_{i-1})]+\frac{1}{\Delta t}\cdot[\mathbf{C}]\cdot\frac{\partial}{\partial\gamma}[\mathbf{A}(t_{i-1})]\quad. \tag{5}$$

Since the stiffness matrix and excitation remain unchanged

$$\frac{\partial}{\partial\gamma}[\mathbf{H}]=[\mathbf{0}],\ \frac{\partial}{\partial\gamma}[\mathbf{I}(t_i)]=[\mathbf{0}],\text{ and }\frac{\partial}{\partial\gamma}[\mathbf{A}(t_i)]=[\mathbf{S}(t_i)], \tag{6}$$

the equation can be reduced to

$$\left([\mathbf{H}]+\frac{1}{\Delta t}\cdot[\mathbf{C}]\right)\cdot[\mathbf{S}(t_i)]=\frac{1}{\Delta t}\cdot[\mathbf{C}]\cdot[\mathbf{S}(t_{i-1})]+\frac{1}{\Delta t}\cdot\frac{\partial}{\partial\gamma}[\mathbf{C}]\cdot[\mathbf{A}(t_{i-1})-\mathbf{A}(t_i)]. \tag{7}$$

In numerical implementation the sensitivity is calculated on the basis of the equations system (7). While the time step remains constant, the matrix of this equations system is the same as (4).

Using this method the sensitivity of all nodal potentials versus conductivity in one finite element is obtained. To obtain the sensitivity versus next element, only new matrix of derivatives should be created. The terms of matrix [**C**] are linear functions of electrical conductivity γ, so the matrix of derivatives contains only constants and zeroes.

To obtain voltage sensitivity of measurement coil, nodal sensitivity has to be integrated over finite element's area of the coil.

3 Tellegen's method

Let us consider a quasi-Poynting vector of the original and adjoint (+) systems

$$\mathbf{S}_p = \frac{1}{2}\int_S \left(\mathbf{E}(t)\cdot\mathbf{H}^+(\tau) + \mathbf{E}^+(\tau)\cdot\mathbf{H}(t)\right)\cdot\mathbf{n}\cdot dS. \tag{8}$$

The Tellegen's method requires an additional term in Maxwell's equations

$$\begin{cases} \operatorname{rot}\mathbf{E}(t) = -\mathbf{L_0}(t) - \mu\cdot\dfrac{\partial\mathbf{H}(t)}{\partial t} \\ \operatorname{rot}\mathbf{H}(t) = \mathbf{J_0}(t) + \gamma\cdot\mathbf{E}(t) + \varepsilon\cdot\dfrac{\partial\mathbf{E}(t)}{\partial t} \end{cases}. \tag{9}$$

$\mathbf{L_0}(t)$ can be interpreted as a magnetic current density. It should be equal zero in every physical system, but the adjoint system can be non-physical. Applying the divergence theorem and substituting equations (9) in (8) yields

$$\int_S \mathbf{E}(t)\times\mathbf{H}^+(\tau)\cdot\mathbf{n}\cdot dS =$$
$$\int_V\left(-\mathbf{H}^+(\tau)\cdot\left(\mu\cdot\frac{\partial\mathbf{H}(t)}{\partial t}+\mathbf{L_0}(t)\right)-\mathbf{E}(t)\cdot\left(\gamma^+\cdot\mathbf{E}^+(\tau)+\varepsilon^+\frac{\partial\mathbf{E}^+(\tau)}{\partial\tau}+\mathbf{J}_0^+(\tau)\right)\right)\cdot dV \tag{10}$$

and

$$\int_S \mathbf{E}^+(\tau)\times\mathbf{H}(t)\cdot\mathbf{n}\cdot dS =$$
$$\int_V\left(-\mathbf{H}(t)\cdot\left(\mu^+\cdot\frac{\partial\mathbf{H}^+(\tau)}{\partial\tau}+\mathbf{L}_0^+(\tau)\right)-\mathbf{E}^+(\tau)\cdot\left(\gamma\cdot\mathbf{E}(t)+\varepsilon\frac{\partial\mathbf{E}(t)}{\partial t}+\mathbf{J_0}(t)\right)\right)\cdot dV. \tag{11}$$

Equations (10) and (11) are also valid for small perturbations of material conductivity $\Delta\gamma$. Neglecting the high order small terms results in following equation

$$\int_V\left(\mathbf{J}_0^+(\tau)\cdot\Delta\mathbf{E}(t)-\mathbf{L}_0^+(\tau)\cdot\Delta\mathbf{H}(t)\right)\cdot dV =$$
$$\int_V \mathbf{E}(t)\cdot\mathbf{E}^+(\tau)\cdot\Delta\gamma\cdot dV + \int_S\left(\mathbf{H}^+(\tau)\times\Delta\mathbf{E}(t)+\mathbf{E}^+(\tau)\times\Delta\mathbf{H}(t)\right)\cdot\mathbf{n}\cdot dS. \tag{12}$$

Above equation determines how to construct the adjoint model. The material parameters are the same, as originals. The boundary conditions should be chosen in such a manner, that the surface integral in (12) vanishes. For this aim the tangential components of **E** and **H** should be equal to zero on the boundary. Both models, original and adjoint, are analyzed in different times t and τ. Usually it will be assumed, that τ is opposite to t: τ = T - t , where T is duration time of the analysis. Integration of equation (12) leads to

$$\int_0^T \int_V \left(\mathbf{J}_0^+(\tau)\cdot \Delta\mathbf{E}(t) - \mathbf{L}_0^+(\tau)\cdot \Delta\mathbf{H}(t)\right)\cdot dV \cdot dt = \int_0^T \int_V \mathbf{E}(t)\cdot \mathbf{E}^+(\tau)\cdot \Delta\gamma \cdot dV \cdot dt \,. \tag{13}$$

Introducing magnetic vector potential into the equation (13)

$$\mathbf{E}(t) = -\frac{\partial \mathbf{A}(t)}{\partial t}, \qquad \mathbf{E}^+(\tau) = -\frac{\partial \mathbf{A}^+(\tau)}{\partial \tau} = \frac{\partial \mathbf{A}^+(\tau)}{\partial t} \tag{14}$$

and assuming that adjoint model is excited by electric current only, Tellegen's equation reduces to

$$\int_0^T \int_V \mathbf{J}_0^+(\tau)\cdot \Delta \frac{\partial \mathbf{A}(t)}{\partial t}\cdot dV \cdot dt = \int_0^T \int_V \frac{\partial \mathbf{A}(t)}{\partial t}\cdot \frac{\partial \mathbf{A}^+(\tau)}{\partial t}\cdot \Delta\gamma \cdot dV \cdot dt \,. \tag{15}$$

Numerical algorithm is based on the above equation. Applying time stepping method

$$t_i = i \cdot \Delta t\,, \quad T = t_n = n \cdot \Delta t \text{ and } \tau_i = t_{n-i+1} = (n-i+1)\cdot \Delta t \tag{16}$$

equation (15) could be shown in matrix form,

$$\sum_{i=1}^{n} \int_V \left[\mathbf{J}_0^+(t_{n-i+1})\right]^T \cdot \Delta \left[\frac{\Delta_t \mathbf{A}(t_i)}{\Delta t}\right]\cdot dV = \sum_{i=1}^{n} \int_{V_e} \left[\frac{\Delta_t \mathbf{A}(t_i)}{\Delta t}\right]^T \cdot \left[\frac{\Delta_t \mathbf{A}^+(t_{n-i+1})}{\Delta t}\right]\cdot \Delta\gamma_e \cdot dV_e \,, \tag{17}$$

where V_e is finite element's area and γ_e is element conductivity versus which the sensitivity is evaluated.

Excitation of the adjoint model depends on the objective function. If the sensitivity is calculated for potential values in mesh nodes, the excitation should be assumed as numerical Dirac's impulse,

$$\mathbf{J}_0^+(\tau)\cdot \Delta V = \delta(\tau_i) = \begin{cases} 1[A] & \text{if} \quad \tau_i = \tau_n \;\Rightarrow\; t_i = t_1 \\ 0[A] & \text{otherwise} \end{cases} \,. \tag{18}$$

Geometric position of this impulse coincides with measurement area ΔV of electromagnetic field. In the simplest case it may be a current introduced into the finite element mesh node. When the measurement proceeds with the help of a coil, the excitation current should be distributed on the area of this coil.

When including excitation (18) into equation (17), the sensitivity equation simplifies considerably,

$$\mathbf{J}_0^+(t_1)\cdot \Delta V \cdot \frac{\Delta}{\Delta\gamma_e}\left(\mathbf{A}(t_n) - \mathbf{A}(t_{n-1})\right) = \frac{1}{\Delta t}\sum_{i=1}^{n} \int_{V_e} \left[\mathbf{A}(t_i) - \mathbf{A}(t_{i-1})\right]^T \cdot \left[\mathbf{A}^+(t_{n-i}) - \mathbf{A}^+(t_{n-i+1})\right]\cdot dV_e \,. \tag{19}$$

The sensitivity versus electrical conductivity will be calculated recurrently as

$$S(t_n) = S(t_{n-1}) + \frac{1}{\Delta t} \cdot \sum_{i=1}^{n} \left(\int_{V_e} [\mathbf{A}(t_i)]^T \cdot [\mathbf{A}^+(t_{n-i})] \cdot dV_e - \int_{V_e} [\mathbf{A}(t_i)]^T \cdot [\mathbf{A}^+(t_{n-i+1})] \cdot dV_e + \right.$$

$$\left. - \int_{V_e} [\mathbf{A}(t_{i-1})]^T \cdot [\mathbf{A}^+(t_{n-i})] \cdot dV_e + \int_{V_e} [\mathbf{A}(t_{i-1})]^T \cdot [\mathbf{A}^+(t_{n-i+1})] \cdot dV_e + \right) \cdot \frac{1}{\mathbf{J}_\mathbf{0}^+(t_1) \cdot \Delta V} \quad . \quad (20)$$

Tellegen's method requires two analyses using FEM in time domain to determine nodal potentials: $[\mathbf{A}(t_i)]$ for original system and $[\mathbf{A}^+(\tau_i)]$ for adjoint system, for every time step. Those values are used to calculate sensitivity accordingly eq. (20). Unlike to method of stiffness and mass matrices derivative when using Tellegen's method, the sensitivity either of one nodal potential or of coil voltage versus conductivity inside every finite element is obtained.

The numerical efficiency of both methods seems to be comparable, but in the case of the large number of measurement points (coils), the method of stiffness and mass matrices derivative will be prefered.

4 Numerical example

A differential probe for eddy-current non-destructive testing has been chosen as a test example. That probe is composed of two exciting coils and measurement coil in between (Fig. 1). The advantage of this probe is the signal equal to zero by crack absence. The calculations were carried out in 2D, so it has been assumed that coils, plate and material cracks are infinitely long. Natural models are three-dimensional, but, apart from difficulties of mathematical formulation of the problem, using three-dimensional model causes difficulties of numerical nature because of the computing power being to low, especially for analyses of vector fields.

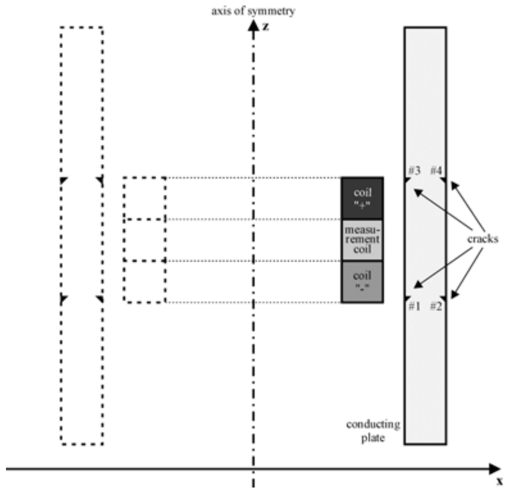
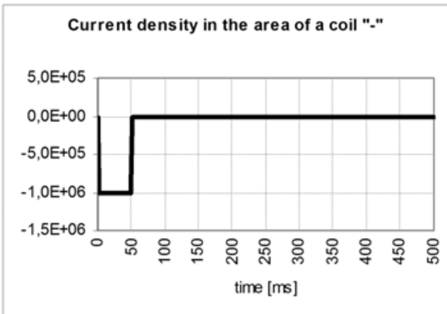


Fig. 1 - *Eddy-current defectometer as two-dimensional model.*

When modelling differential probe, the discretization error in finite elements is of great importance. The mesh should be fully symmetric to keep the differential signal nearby zero without the crack. For the test model shown in Fig. 1 a triangle mesh with 2100 nodes and 4046 elements has been generated.

The voltage induced in the measurement coil is proportional to the magnetic flux. Using magnetic vector potential, the flux is calculated as integral of **A** values at coil's area. The differential coils are excited by rectangular impulse with duration of 50 ms (Fig. 2.). The voltage induced in measurement coil, further called „answer", depends strongly on deep and width of the crack.



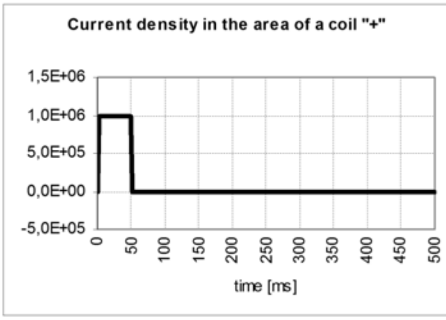


Fig. 2 - *Shape of excitations.*

Table 1
Coil voltage sensitivity values at some time steps t.

t [ms]	$\mathbf{S_D}$ $\left[\frac{V}{S \cdot m}\right]$	$\mathbf{S_S}$ $\left[\frac{V}{S \cdot m}\right]$	$\mathbf{S_T}$ $\left[\frac{V}{S \cdot m}\right]$	$\mathbf{\Delta_{DS}}$ $\left[\frac{V}{S \cdot m}\right]$
1	-1.65842E-22	-2.02295E-22	-2.02295E-22	3.64530E-23
50	-5.75220E-14	-5.74170E-14	-5.74170E-14	-1.05000E-16
100	-1.01710E-13	-1.01770E-13	-1.01770E-13	6.00000E-17
150	3.02800E-14	3.02200E-14	3.02200E-14	6.00000E-17
200	4.38200E-14	4.38200E-14	4.38200E-14	0

$\mathbf{\delta_{DS}}$ [%]	$\mathbf{\Delta_{DT}}$ $\left[\frac{V}{S \cdot m}\right]$	$\mathbf{\delta_{DT}}$ [%]	$\mathbf{\Delta_{ST}}$ $\left[\frac{V}{S \cdot m}\right]$	$\mathbf{\delta_{ST}}$ [%]
21.98%	3.64530E-23	21.98%	0	0%
0.18%	-1.05000E-16	0.18%	0	0%
0.06%	6.00000E-17	0.06%	0	0%
0.20%	6.00000E-17	0.20%	0	0%
0%	0	0%	0	0%

Sensitivity analysis allows to determine the influence of conductivity in single finite element on coil answer. Using two methods described above, sensitivity values in 500 time steps for four differently placed defects (Fig. 1.) have been calculated. First, providing that finite element's conductivity has been decreased by 5%, sensitivity S_D has been determined by means of differential method. In comparison to methods of stiffness and mass matrices derivative (S_S) and Tellegen (S_T), the differential is an approximate method. Sensitivity values have been determined with precision of 6 significant digits. Results obtained by stiffness and mass matrices derivative and Tellegen's method have been identical and tally with results from differential method.

Example results for defects #2 for some time steps have been shown at the Table 1 and sensitivity values as time function for every defect in Fig. 3.

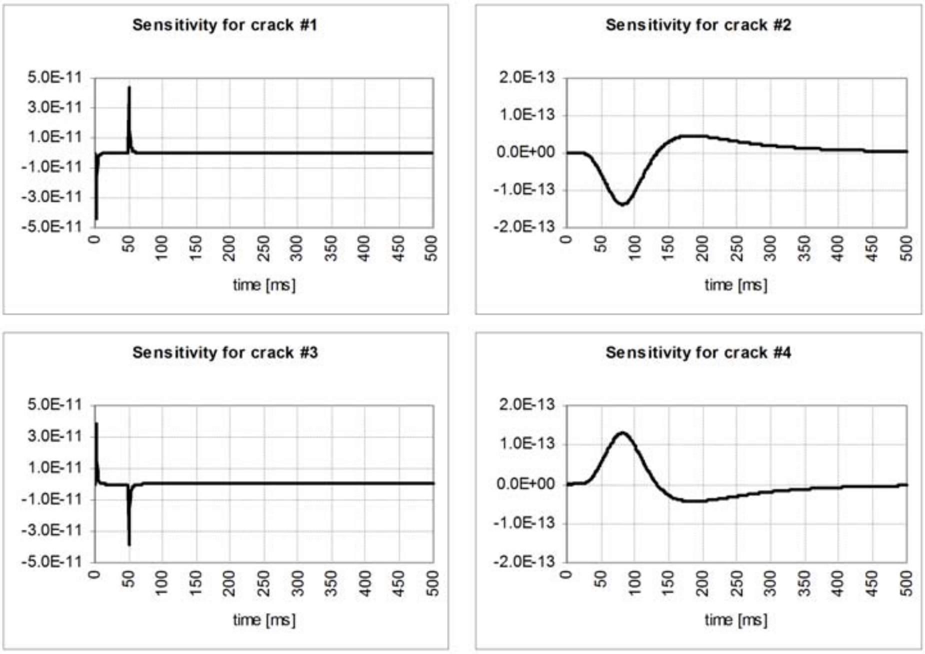


Fig. 3 - *Coil voltage sensitivity as function of time.*

Table 2 shows duration time of analysis on a computer with processor Pentium III-M 850 MHz for one measurement coil. High advantage of Tellegen's method follows the reason, that sensitivities are determined in one computational cycle versus conductivity of all elements.

Table 2
Duration time of analysis.

Differential method	Method of stiffness and mass matrices derivative	Tellegen's method
70 [s]	67 [s]	28 [s]

5 Conclusions

Described methods make sensitivity analysis in time domain possible to carry out. Using eddy-current impulse-method, this is essential for recognition of material's conductivity distribution. Choice of method depends on number of nodes or areas, for which the sensitivity should be determined and on the number of finite elements with varying conductivity, versus which it is calculated. While number of elements is higher Tellegen's method is more optimal, otherwise method of stiffness and mass matrices derivative is useful.

Further research will consist in using gradient algorithm for processing of inverse job on the basis of sensitivity data.

6 References

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