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# Inductance of Cylindrical Coil

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**Abstract:** The cylindrical coreless and bar core coils are used in instrument transformers and many other electromagnetic devices. In the paper, using the separation of variables an analytical formula for the leakage inductance of thin cylindrical coil with unsaturated core is deduced, assuming a simplified path for the frontal flux. The results are compared with well-known experimental data and data resulting from FEM models.

**Keywords:** Cylindrical coil. Inductance. Separation of variables. Bar core transformer.

## 1 Coreless cylindrical solenoid

The first result is the R. Küchler experimentally obtained formula (8), which consider the corresponding equivalent air gap for a thin solenoid with the diameter $d = 2a$ and the length, $l = 2b$ given by the formula [1],

$$\delta = l\left(1+\frac{c}{\alpha}\right); \quad \alpha = \frac{l}{d} = \frac{b}{a} \in (0.2, 20) \Rightarrow c \approx 0.44 . \tag{1}$$

The exact theoretical solution for this case is given by the equation [2]

$$c = \frac{\pi^2}{\Phi(\alpha)} - \alpha ; \quad \Phi(\alpha) = \frac{4\pi}{3}\left[\sqrt{\alpha^2+1}\left(K + \frac{1-\alpha^2}{\alpha^2}E\right) - \frac{1}{\alpha^2}\right]. \tag{2}$$

Here $K$ and $E$ are the elliptic integrals of first and second kind with the module $k$ (annex 1):

$$k = \frac{1}{\sqrt{\alpha^2+1}} . \tag{3}$$

The exact value of inductance of the coreless cylindrical solenoid is

$$L_0 = \frac{\mu_0}{4\pi} w^2 d\,\Phi \quad \text{[H]}. \tag{4}$$

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The dependence of the coefficient $c$ on the solenoid dimensions ratio, calculated with (2) is given in Fig. 1 and agrees with the Küchler experimental result in the specified in [1] domain of α.

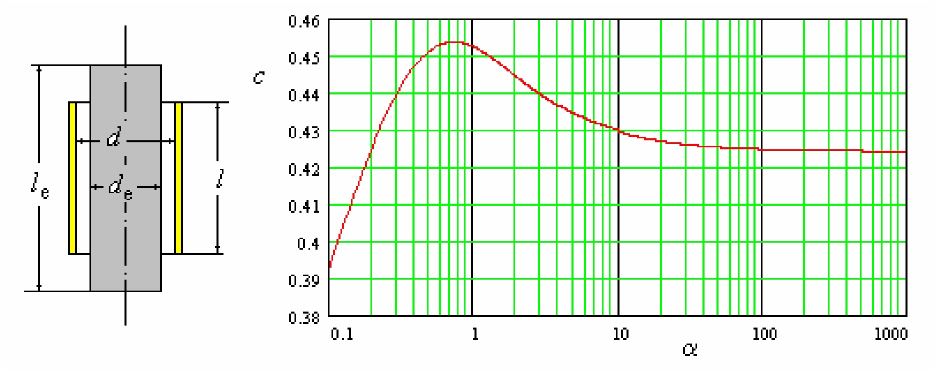


**Fig. 1 -** *Coefficient c for equivalent gap.*

The limits of $c$ for α equal to 0 and infinity can be obtained from (2). For small $k$ the following development of the elliptic integrals are available [3]:

$$K(k)=\frac{\pi}{2}\left[1+\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}\right] \text{ and } E(k)=\frac{\pi}{2}\left[1-\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1}\right]. \quad (5)$$

Considering only two terms in series, for $\alpha \to \infty$ we have the limits

$$\Phi(\alpha)\approx\frac{4\pi}{3}\left\{\frac{\pi}{2}\sqrt{\alpha^2+1}\left[1+\frac{1}{4(\alpha^2+1)}+\frac{1-\alpha^2}{\alpha^2}\left(1-\frac{1}{4(\alpha^2+1)}\right)\right]-\frac{1}{\alpha^2}\right\}=$$

$$\lim_{\alpha\to\infty}\Phi=\frac{\pi^2}{\alpha}-\frac{4\pi}{3\alpha^2} \qquad \Rightarrow c(\infty)\equiv\lim_{\alpha\to\infty}\left(\frac{\pi^2}{\Phi(\alpha)}-\alpha\right)=\frac{4}{3\pi}\approx 0.424. \qquad (6)$$

For $\alpha \to 0$ [2]

$$\Phi(\alpha)\approx 2\pi\left[\left(1+\frac{\alpha^2}{8}-\frac{\alpha^4}{64}+\ldots\right)\ln\frac{4}{\alpha}-\frac{1}{2}+\frac{\alpha^2}{32}+\frac{\alpha^4}{96}+\ldots\right]\to\infty. \qquad (7)$$

Küchler [1] proposes to calculate the inductance of bar core cylindrical coil, considering the solenoid length equal to zero and the equivalent air gap $\delta \approx c\,d$ and introduces the correction coefficients $k_1$ and $k_2$, which take into consideration the core length $l_e = 2h$ and core diameter $d_e$. For $l_e - l \in (2...3)\,d_e$ the factor $k_2 \approx \frac{d_e}{d}$. This means that, in this case, in the next formula (8) the coil diameter can be simple substitute by core diameter $d_e$

$$L_e = \frac{\mu_0 w^2 \pi d}{4c} k_1 k_2 ; \quad k_1 = \begin{cases} 1 + 0.1\dfrac{b}{a} & h = b \\ 1 - 0.5\dfrac{b}{a} + 0.6\dfrac{h}{a} & h > a \end{cases} ; \quad k_2 \approx \frac{d_e}{d} . \tag{8}$$

Here $d$ is the coil diameter and $d_e$ is the core diameter.

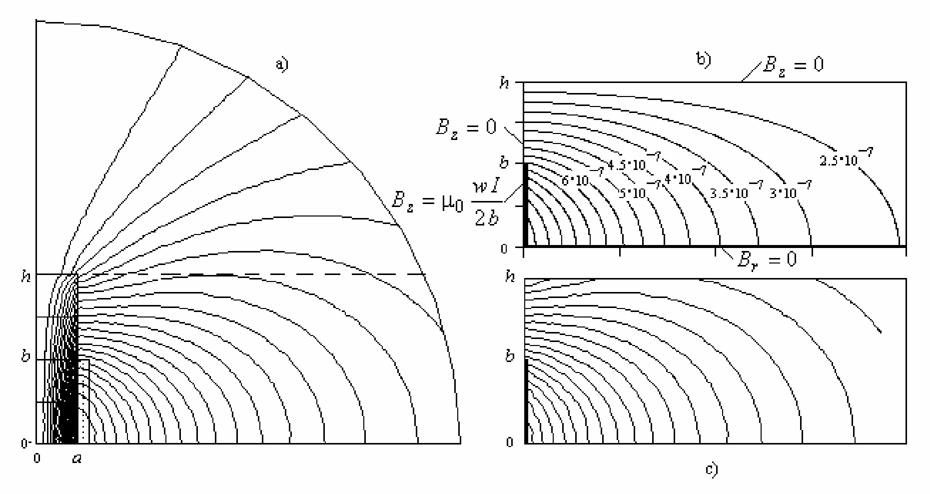


**Fig. 2 -** *Magnetic field pattern: a), c) - calculated wit FEM; b) calculated with (18).*

## 2 Leakage inductance of bar core solenoid

### 2.1 Field problem

We will consider a symmetrical thin solenoid closely fitting to cylindrical steel core.

The vector magnetic potential in D = $[r > a] \times [0 < z < h]$ will satisfy the equation:

$$\mathrm{div}\, \boldsymbol{A} = 0 \quad \Rightarrow \quad \mathrm{rot}\,\mathrm{rot}\, \boldsymbol{A} = -\Delta \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

and

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} - \frac{A}{r^2} + \frac{\partial^2 A}{\partial z^2} = \begin{cases} -\mu_0 \boldsymbol{J}\, \delta(r - a, z), & |z| \le h \\ 0 & r > a \end{cases} . \tag{9}$$

The magnetic flux line issuing from the core edge will be considered a horizontal straight line (dashed line in Fig. 2) and, consequently, we can approximate the boundary conditions as follows:

$$B_r\big|_{z=0}=0;\;\; B_r\big|_{z=h}=0;\;\; H_z\big|_{r=a}=\begin{cases}\dfrac{wI}{2b(1+\eta)}, & |z|<b\\ 0, & |z|>b\end{cases} \text{ and } A\big|_{r=\infty}=0\,. \quad (10)$$

Here $w$ is the coil number of turns and $\eta\approx 0$ the ratio of the internal to external tangential magnetic field on coil surface.

Using the separation of variables method and considering $A(r,z)=R(r)Z(z)$ we will obtain the equations:

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}-\frac{1}{r^2}=-\frac{Z''}{Z}=\lambda_i^2 \qquad x=\lambda_i\, r\Rightarrow$$

$$\frac{\partial^2 R}{\partial x^2}+\frac{1}{x}\frac{\partial R}{\partial x}-\left(1+\frac{1}{x^2}\right)R=0 \quad\Rightarrow R(x)=C_1\,K_1(x)+C_2\,I_1(x);$$

$$\frac{\partial^2 Z}{\partial z^2}+\lambda_i^2\,Z=0 \qquad \Rightarrow Z(z)=C_3\cos(\lambda_i\, z)+C_4\sin(\lambda_i\, z)\,. \quad (11)$$

Here $K_1$ and $I_1$ are the modified Bessel functions of first order.

Taking into account the boundary conditions for $z = 0$ and $r = \infty$ which give $C_2 = C_4 = 0$, we will seek for the solution of the above problem in the following form, which satisfy the boundary condition for $z = h$,

$$A(r,z)=\mu_0\, wI\sum_i A_i\, K_1(\lambda_i\, r)\cos(\lambda_i\, z)+\frac{C}{r};\quad \lambda_i=\frac{\pi}{h}\left(\frac{1}{2}+i\right). \quad (12)$$

Using known equation for Bessel functions [3],

$$K_1'(x)+\frac{1}{x}K_1(x)=-K_0(x) \text{ and } K_0'(x)=-K_1(x)$$

the components of flux density will be:

$$B_r(r,z)=-\frac{\partial A}{\partial z}=\mu_0\, wI\sum_{i=0}^{\infty}\lambda_i A_i\, K_1(\lambda_i\, r)\sin(\lambda_i\, z)$$

and

$$B_z(r,z)=\frac{A}{r}+\frac{\partial A}{\partial r}=-\mu_0\, wI\sum_{i=0}^{\infty}\lambda_i A_i\, K_0(\lambda_i\, r)\cos(\lambda_i\, z)\cdot \quad (13)$$

The boundary condition for $r = a$ determines the $A_i$ coefficients. Equating (9) and (13) for $\eta\approx\infty$ we obtain:

$$wI\sum_{i=0}^{\infty}\lambda_i\, A_i\, K_0(\lambda_i\, a)\cos(\lambda_i\, z)=H_z\big|_{r=a}=\begin{cases}\dfrac{wI}{2b}, & |z|<b\\ 0, & |z|>b\end{cases}. \quad (14)$$

Multiplying the last equation by $\cos(\lambda_i z)$ and integrating between 0 and $h$, it results

$$\int_0^h \cos^2(\lambda_i z)\mathrm{d}z = \frac{h}{2} \quad \Rightarrow A_i = \frac{\sin(\lambda_i b)}{\lambda_i^2 b h K_0(\lambda_i a)}. \tag{15}$$

## 2.2 Frontal flux

We will approximate the magnetic field pattern for $|z| > h$ with the current lines in point contact. Under such assumptions the magnetic flux passing through the end of the $2a$ diameter cylindrical core and the corresponding inductance, can be evaluated as in [4]

$$\Phi_1 = 2\mu_0 w I a \quad \Rightarrow L_1 = \frac{w\Phi_1}{I} = 2\mu_0 w^2 a. \tag{16}$$

This assumption determines the value of the vector magnetic potential $A(a, h)$ and the constant $C$:

$$\Phi_1 = 2\pi a A(a,h) = 2\pi C \quad \Rightarrow C = \frac{\mu_0}{\pi} w I a. \tag{17}$$

For $r > a$, the vector magnetic potential will be

$$A(r,z) = \mu_0 w I \left[\frac{a}{r} + \sum_{i=0}^{\infty} A_i K_1(\lambda_i r)\cos(\lambda_i z)\right] \quad r > a, \quad |z| < h. \tag{18}$$

The equations of magnetic flux lines are $r A(r,z) = \text{const.}$ In Fig. 2 b) and c) we can see the enough good similarity of the thus calculated field patterns with the obtained by FEM.

## 2.3 Lateral flux

The magnetic flux emergent from the cylindrical part of the core above the coil and the corresponding inductance will be

$$\Phi_2 = 2\pi a\left[A(a,b) - A(a,h)\right] = 2\pi\mu_0 w I a \sum_{i=0}^{\infty} A_i K_1(\lambda_i a)\cos(\lambda_i b)$$

and

$$L_2 = 2\pi\mu_0 w^2 a \sum_{i=0}^{\infty} A_i K_1(\lambda_i a)\cos(\lambda_i b). \tag{19}$$

The interlinkage through the coil and corresponding inductance will be:

$$\Psi_3 = 2\pi a \frac{w}{b}\int_b^0 z B_r(a,z)\mathrm{d}z = 2\pi\mu_0 w^2 I \frac{a}{b}\sum_{i=0}^{\infty} A_i K_1(\lambda_i a)\lambda_i \int_0^b z\sin(\lambda_i z)\mathrm{d}z \tag{20}$$

and

$$L_3 = 2\pi\mu_0\, w^2\, a \sum_{i=0}^{\infty} A_i\, K_1(\lambda_i a)\left[\frac{\sin(\lambda_i\, b)}{\lambda_i b} - \cos(\lambda_i\, b)\right]. \tag{21}$$

### 2.4 Total inductance

The total inductance of cylindrical coil closely fitting to cylindrical core will be

$$L = L_1 + L_2 + L_3 = 2\pi\mu_0\, w^2\, a\left[\frac{1}{\pi} + \sum_{i=0}^{\infty} A_i\, K_1(\lambda_i a)\frac{\sin(\lambda_i\, b)}{\lambda_i b}\right]. \tag{22}$$

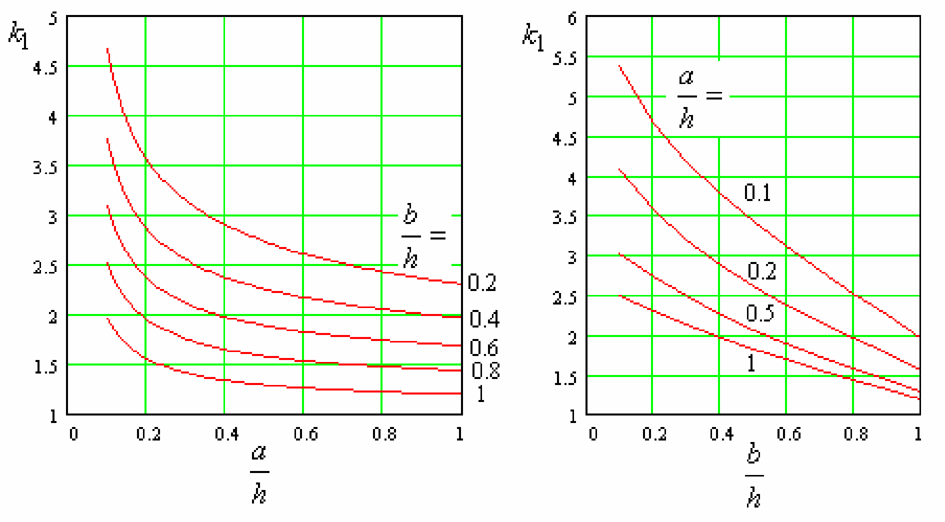


**Fig. 3** - *Coefficient $k_1$ from formula (8), calculated with (24).*

The Küchler coefficient $k_1$ from (8) defined as ratio of the core coil inductance (22) to the zero length coreless coil inductance is now

$$k_1 = \frac{L}{L_0\big|_{l=0}} = 4\left(\frac{\pi^2}{\Phi(\alpha)} - \alpha\right)\left[\frac{1}{\pi} + \sum_{i=0}^{\infty} A_i\, K_1(\lambda_i a)\frac{\sin(\lambda_i\, b)}{\lambda_i b}\right]. \tag{23}$$

Taking into account the equations (12) and (15) we obtain for $k_1$ coefficient the equations:

$$k_1(x,y) = 4\left(\frac{\pi^2}{\Phi\left(\frac{y}{x}\right)} - \frac{y}{x}\right)\left[\frac{1}{\pi} + \sum_{i=0}^{\infty} T_i\right]; \quad x = \frac{a}{h};\ y = \frac{b}{h}; \quad j = i + \frac{1}{2}$$

$$T_i = \frac{K_1(\pi j x)}{K_0(\pi j x)}\frac{\sin^2(\pi j y)}{(\pi j)^3 y^2}; \quad R_n\big|_{x>0.01} = \sum_{i=n}^{\infty} T_i < \frac{1}{2\pi^3}\left(\frac{h}{b}\right)^2 \frac{1+\frac{15}{n+1}}{n^2}. \qquad (24)$$

The values of $k_1$, calculated with ten terms ($R_n < 0.007$, for $b > 0.2h$) in series (24), are given in Fig. 3 (annex 2).

## 3 Comparison with experimental and FEM results

The comparison with Küchler's formula (8) and the results obtained by FEM, considering asymptotic boundary conditions $\frac{\partial A}{\partial r} + \frac{A}{r} = 0$ on the outer radius of the region $r_0 = 0.1$ m, for $a = 1$ cm, is given in Fig. 4.

A better agreement of theoretical results than of given in [1], with obtained by FEM can be observed in Fig. 4, especially for short solenoid and long core.

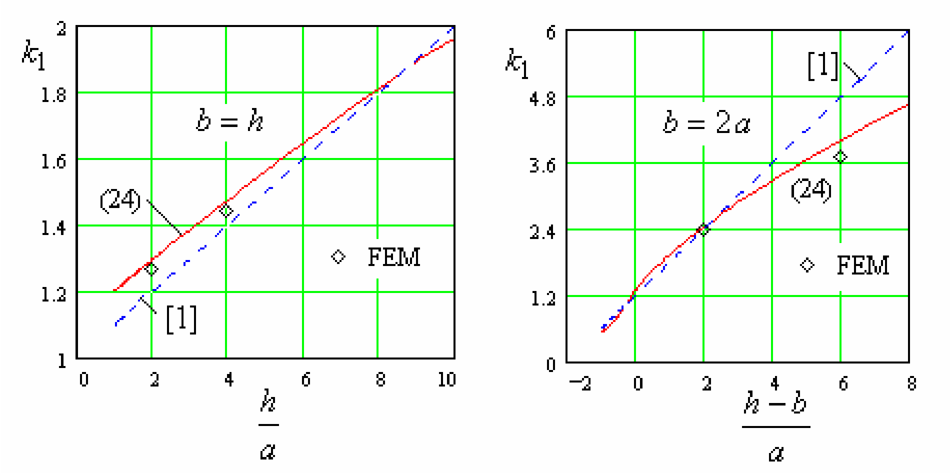


**Fig. 4 -** *Comparison of the calculated with (24) $k_1$ values with the given in [1] and determined using FEM.*

## 4 Conclusion

1. The inductance of coreless cylindrical coil can be always calculated with formula (4) with the coefficient *c* calculated with (2) or given in Fig. 1 for wide range of α.

2. The inductance of cylindrical coil can be calculated for small diameters, using the Küchler's experimental formula (8) and the given in [1] curves for $k_1$. For large diameters the given in [5] curves also can be used.
3. For all the diameters and for long or short core, the inductance of thin cylindrical coil, closely fitting to cylindrical unsaturated steel core, can by evaluated using the proposed formula (22) or the Küchler's experimental formula (8) and the curves from Fig. 3 for $k_1$.
4. The inductances obtained with the proposed formula (22) or with Küchler's formula (8) with $k_1$ from Fig. 3 are in better agreement with the FEM results, with asymptotic boundary conditions.

# 5 Acknowledgement

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**Annex 1**

Elliptic int m = k$^2$

$$K_0(k) = \frac{\pi}{2}\left[1 + \sum_{n=1}^{10}\left[\frac{\left[\prod_{i=1}^{n}(2i-1)\right]}{2^n n!}\right]^2 k^{2n}\right], \quad E_0(k) = \frac{\pi}{2}\left[1 - \sum_{n=1}^{10}\left[\frac{\left[\prod_{i=1}^{n}(2i-1)\right]}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1}\right],$$

$$K_1(m) = \left(1.3862944 + 0.1119723m + 0.0725296m^2\right) - \left(0.5 + 0.1213478m + 0.0288729m^2\right)\ln(m),$$

$$E_1(m) = \left(1 + 0.4630151m + 0.1077812m^2\right) - \left(0.2452727m + 0.0412496m^2\right)\ln(m),$$

$$M = 0.99, \quad K(m) = \text{if}\left(m < M, K_1(m), K_0\left(\sqrt{1-m}\right)\right), \quad E(m) = \text{if}\left(m < M, E_1(m), E_0\left(\sqrt{1-m}\right)\right).$$

**Annex 2**

For $x = \frac{a}{h} > 0.01$ exists the following inequality:

$$\frac{K_1\left(\pi(i+05)x\right)}{K_0\left(\pi(i+05)x\right)} < 1 + \frac{15}{i+1}. \tag{A2.1}$$

Replacing in (24) we obtain:

$$R_n\big|_{x>0.01} = \sum_{i=n}^{\infty} T_i < \frac{1+\frac{15}{n+1}}{\pi^3 y^2}\sum_{i=n}^{\infty}\frac{1}{(i+0.5)^3} \approx \frac{1}{\pi^3}\left(\frac{h}{b}\right)^2\left(1+\frac{15}{n+1}\right)\int_{x=n}^{\infty}\frac{dx}{x^3}. \tag{A2.2}$$

From which results the rest evaluation given in (24).

## 6 References

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