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New Approach to Far Field Analysis for Radiation Pattern Estimation Using FDTD Method

Bojana Nikolić[1], Bojan Dimitrijević[1], Slavoljub Aleksić[1], Nebojša Raičević[1], Nenad Milošević[1]

Abstract: In this paper an approach to efficient computation of radiation pattern in FDTD simulation environment is presented. A necessary large distance from the radiating object is achieved by multigrid space discretization with unilaterally connected subdomains. A numerical dispersion is reduced using more general complex-envelope finite difference time domain (CE-FDTD) formulation and high order accuracy FDTD schemes where possible. In order to examine how much the introduced algorithm complexity and increased demands concerning computational power and memory are justified by the gain in accuracy, several different scenarios were considered.

Keywords: Finite Difference Time Domain, Multigrid, Radiation Pattern, Maxwell's Equations.

1 Introduction

Following the rapid development of computer technology today, finite difference time domain (FDTD) method became a primary available tool for fundamental understanding, analysis and characterization of wide range of electromagnetic problems, starting from antennas, microwave wave circuits, electromagnetic compatibility (EMC) issues, bioelectromagnetics, electromagnetic scattering up to novel materials and nanophotonics. In many cases, there is a crucial need to simulate a large domain in space, while the accuracy of the obtained results should be maintained. One of such applications is when a direct use of FDTD is a method of choice for obtaining radiation pattern (instead of commonly used near-to-far field transformation). If temporal and spatial discretization in uniform FDTD grid is performed sufficiently fine in order to achieve stability and keep the numerical dispersion low, consumption of memory and computational power increases unacceptably. These opposite demands can be fulfilled by an FDTD algorithm with multiple grid regions.

[1]University of Niš, Faculty of Electronic Engineering, Aleksandra Medvedeva 14, 18000 Niš, Serbia;
E-mails: bojana.nikolic@elfak.ni.ac.rs; bojan.dimitrijevic@elfak.ni.ac.rs; bojan.dimitrijevic@elfak.ni.ac.rs; slavoljub.aleksic@elfak.ni.ac.rs; nebojsa.raicevic@elfak.ni.ac.rs; nenad.milosevic@elfak.ni.ac.rs

A multiple subgridding in FDTD has been extensively studied in the literature. Some of the earliest papers regarding this subject reported sequential computations [1] and subgridding in space only (graded mesh) [2] as possible solutions to the mesh refinement. However, the numerical dispersion shown to be highly dependent on the mesh density and the time step must correspond to the smallest grid throughout the computational space.

Considering the solutions that include subgridding in space and time, the most of the proposed algorithms are based on use of the appropriate interpolation/extrapolation schemes in order to determine field components on the boundary between fine and coarse grid [3 – 6], [7 – 8]. In the mesh refinement algorithm (MRA) [5] and the variable step-size method (VSSM) [4] a discretized wave equation is used for calculation of missing field values (update of field components) at the transition region. However, in VSSM required spatial difference terms in the fine grid region are obtained by interpolating the field values from the coarse grid, while in MRA they are obtained by interpolating the coarse-grid spatial differences. Unfortunately, many of the subgridding methods belonging to this class are not provably stable, and for many cases, they exhibit late-time instabilities [8]. Another disadvantage is low ratio of the space steps of the grids, due to unavoidable spurious reflections on the surface between two grids.

In [9] Bérenger proposes Huygens subgridding (HSG) where the physical connection between two grids is realized by means of Huygens surfaces. In one grid the other grid is viewed through equivalent currents and the problem is reduced to the physical one. The advantages of the method are arbitrarily large ratio of the space steps (but always an odd number) and no significant reflections on the interface between the grids. The drawback is the presence of a late time instability, which can be reduced, though, by filtering the highest frequencies of the currents.

A method that attracted a significant attention in the literature in recent years is a wavelet-based time-domain method, referred to as multiresolution time-domain (MRTD) [10]. According to this method, the field components are spatially expanded in a basis composed of scaling and wavelet functions and pulse functions are used for the temporal field expansion. Galerkin method is then applied to derive finite difference equations with respect to the field scaling and wavelet spatiotemporal coefficients [11]. Depending on the wavelet basis that is used, several different types of MRTD algorithm are developed ([10]-[13] and references therein). Among them, Haar MRTD is the most frequently employed, because of its simplicity and certain advantages regarding parallelization and domain decomposition.

In [14 – 16] hybrid techniques that apply FDTD in large volumes, combined with finite-element method (FEM) near complex boundaries are proposed, since FEM allows good approximations of complex boundaries and with edge elements. Some of those combined techniques suffer from late time instabilities [15 – 16].

Employment of high order accuracy algorithms is an efficient way to reduce dispersion error. This is especially important when a multigrid computational domain is used. Namely, the utilization of high order accuracy algorithms in coarser grid in multigrid FDTD formulation leads to satisfactory level of accuracy and yet reduced computational time and significant computational power and memory savings. A comprehensive stability and numerical dispersion analysis of higher order finite difference schemes can be found in [17]. The main difficulties of this approach are related to boundary conditions and modeling of discontinuities.

A complex-envelope FDTD (CE-FDTD) scheme [18] is a general FDTD formulation that can be important tool to combat the numerical dispersion problem, which is especially a concerning issue in multiple grid algorithms. Using this method, the signal can be sampled in accordance with the bandwidth of the signal rather than its maximum frequency, which yields the time step significantly increased in comparison with the conventional FDTD. A comprehensive stability and numerical dispersion analysis of CE-FDTD can be found in [19]. Although the analyses showed that the introduced complexity was not quite justified by the obtained gain in maximum time step size, there is still a strong motivation to employ this formulation, since in combination with high order accuracy FDTD approaches, the expected reduction of numerical dispersion might be achieved.

One of the most efficient techniques for termination of computational domain in FDTD is the convolutional perfectly matched layer (CPML) [20]. It is based on the stretched coordinate form [21] and the use of complex frequency shift (CFS) of PML parameters [22]. It is shown that the CFS-PML is highly absorptive of evanescent modes and can provide significant memory savings when computing the wave interaction of elongated structures, sharp corners, or low frequency excitations [20]. The conventional CPML cannot be directly applied in CE-FDTD environment, but with the adjustments presented in [23] it can be used for termination of CE-FDTD computational domain.

In this paper an approach to efficient computation of radiation pattern in FDTD simulation environment is presented. A necessary large distance from the radiating object is achieved by multigrid space discretization with unilaterally connected subdomains. Namely, in outer coarse domain, inner fine domain is considered as equivalent source which is defined through the corresponding field distribution on its surface. On the other hand, outer domain has no influence to the inner domain, which is a reasonable constrain, since the main application of such a system is to obtain far field and radiation pattern of the antenna. In this way convergence and stability of the algorithm are guaranteed and equal to the case of conventional FDTD algorithm with uniform grid. A numerical dispersion is reduced using more general CE-FDTD (complex-envelope finite difference time domain) formulation and high order accuracy FDTD schemes where possible. In order to examine how much the introduced algorithm complexity and increased demands concerning computational power and memory are justified by the gain in accuracy, several different scenarios were considered.

2 FDTD Formulation

When applying FDTD algorithm, Maxwell's curl equations are first converted to the corresponding scalar partial differential equations (PDEs). The temporal discretization of PDEs is usually done by second-order accurate two-point central difference method. The spatial discretization is usually done applying second-order accurate method as well, but in order to achieve better accuracy, spatial derivatives can be approximated using higher order accuracy schemes. In practice, schemes of accuracy order up to six are considered, since the obtained gain of higher accurate schemes does not justify their complexity.

In this paper CE-FDTD algorithm will be briefly presented, as one general FDTD formulation. A detailed description of CE-FDTD algorithm can be found in [24 – 25]. A signal can be presented as

$$\left(H_\nu, E_\nu\right) = \mathrm{Re}\{(\hat{H}_\nu, \hat{E}_\nu)e^{j\omega_c t}\}, \tag{1}$$

where ω_c is the carrier frequency and operator $\mathrm{Re}\{\}$ returns the real part of a complex number. $\hat{H}_\nu$ and $\hat{E}_\nu$ denote associated complex-envelope representations

$$(\hat{H}_\nu, \hat{E}_\nu) = (H_{p\nu}, E_{p\nu}) + j(H_{q\nu}, E_{q\nu}), \tag{2}$$

Magnitudes $(H_{p\nu}, E_{p\nu})$ and $(H_{q\nu}, E_{q\nu})$ are in-phase and quadrature parts of $(\hat{H}_\nu, \hat{E}_\nu)$.

Using (1), CE Maxwell's equations can be obtained as

$$\varepsilon_x \frac{\partial \hat{E}_x}{\partial t} + \hat{\sigma}_{ex}\hat{E}_x = \frac{\partial \hat{H}_z}{\partial y} - \frac{\partial \hat{H}_y}{\partial z}, \tag{3}$$

$$\mu_x \frac{\partial \hat{H}_x}{\partial t} + \hat{\sigma}_{mx}\hat{H}_x = \frac{\partial \hat{E}_y}{\partial z} - \frac{\partial \hat{E}_z}{\partial y}, \tag{4}$$

where $\hat{\sigma}_{ex} = \sigma_{ex} + j\omega_c$ and $\hat{\sigma}_{mx} = \sigma_{mx} + j\omega_c$. Electric and magnetic conductivity are denoted as σ_{ex} and σ_{mx}, respectively. It should be mentioned that the magnetic conductivity σ_{mx} is, strictly speaking, not physically realistic, but is often used to describe magnetic losses [26] and it will be included in this analysis. This term was labeled as σ_{mx} in order to maintain similar notation in Faraday's and Ampere's law. For the brevity only equations for $\hat{E}_x$ and $\hat{H}_x$ field components are presented.

Applying the same techniques for spatial and temporal discretization as in conventional FDTD formulation (second-order accurate two-point central difference technique and time-average time approximation), one can obtain CE-FDTD update equations. For $\hat{H}_x$ and $\hat{E}_x$ field components, it is

$$\begin{aligned} \hat{H}^{n+1/2}_{x(i,j+1/2,k+1/2)} &= a_{x,H}\hat{H}^{n-1/2}_{x(i,j+1/2,k+1/2)} + \\ &+ b_{x,H}\sum_{r=0}^{N/2-1} c_r \left(\frac{\hat{E}^n_{y(i,j+1/2,k-r+1)} - \hat{E}^n_{y(i,j+1/2,k-r)}}{\Delta z} - \right. \\ &\left. - \frac{\hat{E}^n_{z(i,j-r+1,k+1/2)} - \hat{E}^n_{z(i,j-r,k+1/2)}}{\Delta y} \right), \end{aligned} \tag{5}$$

$$\begin{aligned} \hat{E}^{n+1}_{x(i+1/2,j,k)} &= a_{x,E}\hat{E}^{n}_{x(i+1/2,j,k)} + \\ &+ b_{x,E}\sum_{r=0}^{N/2-1} c_r \left(\frac{\hat{H}^{n+1/2}_{z(i+1/2,j-r+1/2,k)} - \hat{H}^{n+1/2}_{z(i+1/2,j-r-1/2,k)}}{\Delta y} - \right. \\ &\left. - \frac{\hat{H}^{n+1/2}_{y(i+1/2,j,k-r+1/2)} - \hat{H}^{n+1/2}_{y(i+1/2,j,k-r-1/2)}}{\Delta z} \right), \end{aligned} \tag{6}$$

$$\hat{H}^n_u = \hat{H}^n_{u(i,j,k)} = \hat{H}_u\left(i\Delta x, j\Delta y, k\Delta z, n\Delta t\right), \tag{7}$$

$$\hat{E}^n_u = \hat{E}^n_{u(i,j,k)} = \hat{E}_u\left(i\Delta x, j\Delta y, k\Delta z, n\Delta t\right), u \in \{x, y, z\}, \tag{8}$$

where Δt is the time step, Δx, Δy and Δz are spatial steps along x, y and z axis, respectively. Update coefficients $a_{v,E}$, $b_{v,E}$, $a_{v,H}$ and $b_{v,H}$ ($v = x, y, z$) are given in **Table 1**. In **Table 2** higher order accuracy coefficients c_r are presented for accuracy order $N = 2$, 4, 6 and 8. It should be mentioned that the use of higher order accuracy schemes ($N = 4$, 6 or 8) is limited to those cells which share the same properties with the neighbouring cells required in update equations.

Table 1
Update Coefficients CE-FDTD [27].

Update Coefficients	
$a_{v,H}$	$\dfrac{1-\hat{\sigma}_{mv}\Delta t/(2\mu_v)}{1+\hat{\sigma}_{mv}\Delta t/(2\mu_v)}$
$b_{v,H}$	$\dfrac{\Delta t/\mu_v}{1+\hat{\sigma}_{mv}\Delta t/(2\mu_v)}$
$a_{v,E}$	$\dfrac{1-\hat{\sigma}_{ev}\Delta t/(2\varepsilon_v)}{1+\hat{\sigma}_{ev}\Delta t/(2\varepsilon_v)}$
$b_{v,E}$	$\dfrac{\Delta t/\varepsilon_v}{1+\hat{\sigma}_{ev}\Delta t/(2\varepsilon_v)}$

Table 2
High Order Coefficients in FDTD.

High Order Coefficients	c_r			
r	(2,2)	(2,4)	(2,6)	(2,8)
0	1	1.25	1.171875	1.1962890625
1	-	-0.0416666667	-0.0651041667	-0.0797526
2	-	-	0.0046875	0.0095703125
3	-	-	-	-0.000697544643

3 CPML Algorithm

The CPML implementation, based on the conventional FDTD formulation, along with undertaken code optimizations, is described in detail in [28]. The modified CPML algorithm for termination of CE-FDTD computational domain is presented in [23].

4 Subgridding Algorithm

In multigridding approach that we propose a computational domain is consisted of two (in general case *N*) concentric subdomains with different mesh density. Each of the subdomains is terminated by its CPML boundaries and represents totally independent system which fulfills its own CFL criterion. It is assumed that there are no discontinuities of material properties on the boundary between two domains. The interaction of the subdomains is unilateral. This constrain seems to be justified since the main application of such a system is to obtain far field and radiation pattern of the antenna. Namely, outer coarse domain has no influence to the inner fine domain. On the other hand, in outer coarse domain, inner fine domain is considered only through the assigned field values in particular nods, which is substantially equivalent to the hard source excitation of the outer domain. In this way convergence and stability of the algorithm are guaranteed and equal to the case of conventional FDTD algorithm with uniform grid.

5 Numerical Results

A multigrid computational domain, as well as high-order-in-space algorithms has been implemented as operating modes available in actual realized FDTD simulation environment. Also the choice between the classical FDTD formulation and CE-FDTD formulation is available. In order to examine how much the introduced algorithm complexity and increased demands concerning computational power and memory are justified by the gain in accuracy, several different scenarios were considered. For the purpose of illustration, a well-known example of a half-wave dipole antenna is presented. A sine excitation signal at 2.4GHz is used. The number of used CPML layers in both subdomains is 8.

In Fig. 1 the radiation pattern of a dipole antenna for the case of standard second-order-in-time second-order-in-space finite difference formulation (FDTD(2,2)) is presented. A multigrid computational domain with two subdomains is applied. The dimensions of the inner and outer subdomain are $100\times100\times100$ and $200\times200\times200$, respectively. The refinement mesh factor is set to be 7. The use of unilaterally connected computational subdomaines does not produce significant undesireable reflections or disturbance of any other kind. Deviations of the pattern from the expected form that can be noticed originate from the very high level of present numerical dispersion. The same behavior can be observed in 3D representation in Fig. 2 for the described test case.

In Fig. 3 the radiation pattern of a dipole antenna for the case of standard second-order-in-time fourth-order-in-space finite difference formulation (FDTD(2,4)) is presented. A multigrid computational domain with two subdomains is also applied. The dimensions of the inner and outer subdomain are $100\times100\times100$ and $200\times200\times200$, respectively. The refinement mesh factor is 7. The obtained radiation pattern is more closely corresponding to the theoretically expected form. The introduction of the fourth-order-in-space finite difference formulation reduced the dispersion error significantly. The same behavior can be observed in 3D representation in Fig. 4 for the described test case.

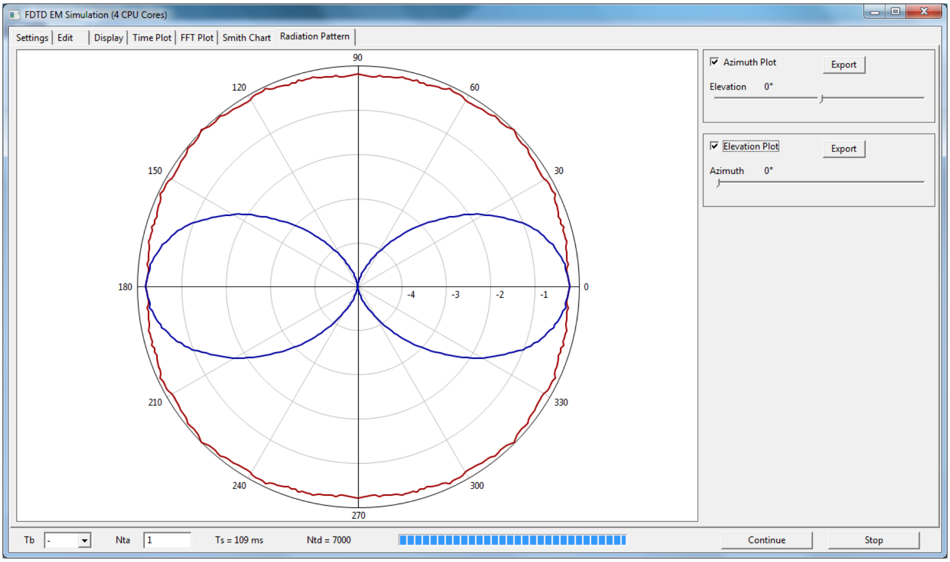

Fig. 1 – *Radiation pattern of dipole antenna for the case of* FDTD(2,2) *algorithm.*

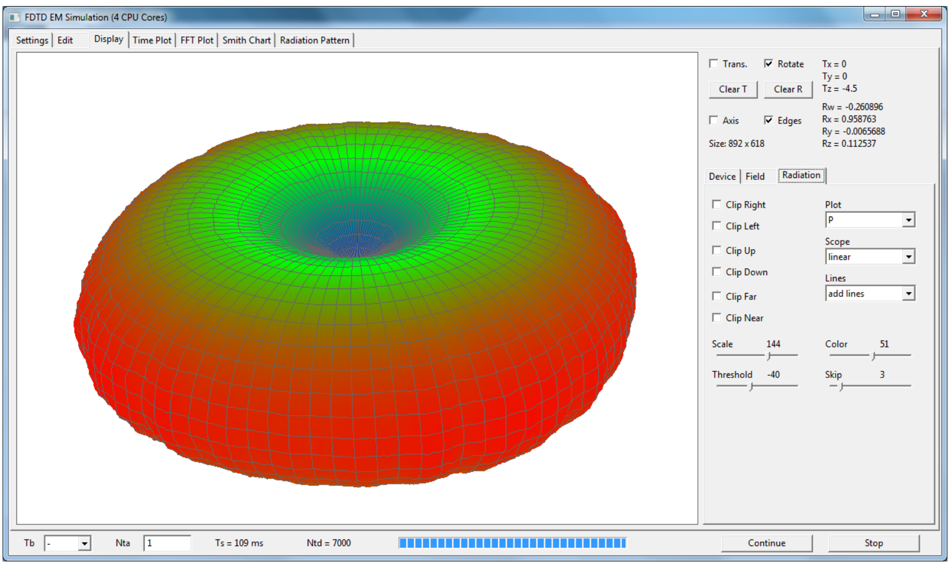

Fig. 2 – 3D *radiation pattern of dipole antenna for the case of* FDTD(2,2) *algorithm.*

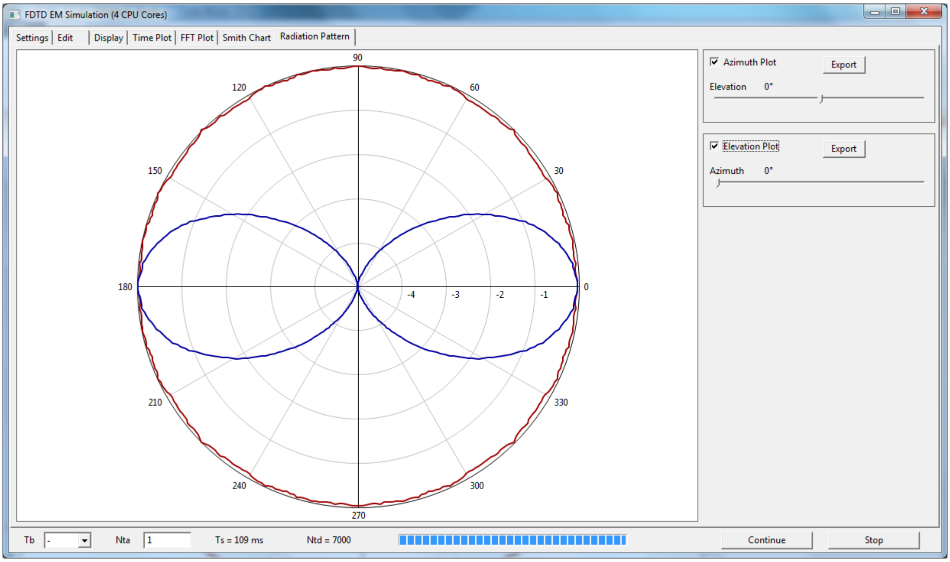

Fig. 3 – *Radiation pattern of dipole antenna for the case of* FDTD(2,4) *algorithm.*

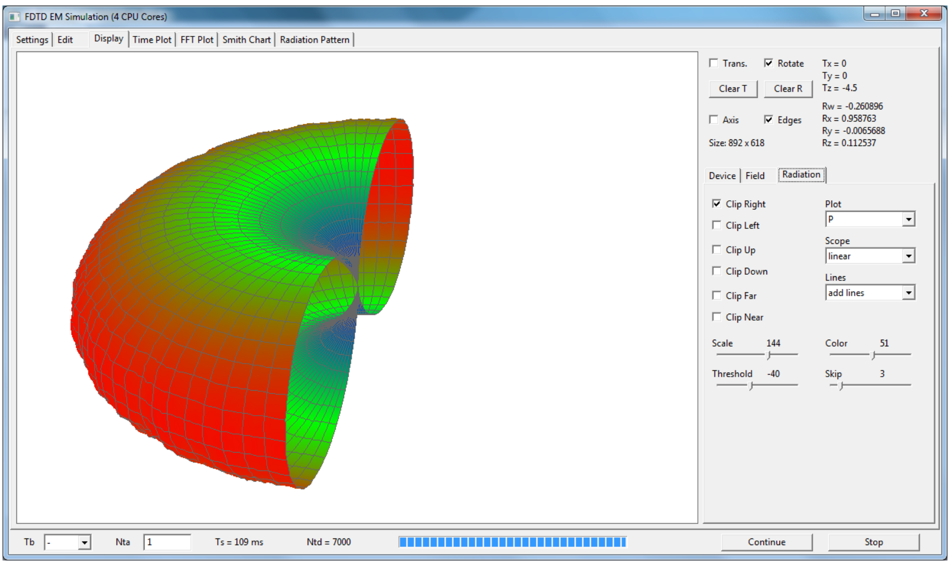

Fig. 4 – *3D radiation pattern of dipole antenna with cross section view for the case of* FDTD(2,4) *algorithm.*

In Fig. 5 three groups of radiation pattern curves are comparatively presented: a- second-order-in-time second-order-in-space FDTD and CE-FDTD formulations; b- second-order-in-time fourth-order-in-space FDTD and CE-FDTD formulations; c- second-order-in-time sixth-order-in-space FDTD and CE-FDTD formulations. A multigrid computational domain with two subdomains is applied in all cases. It can be noticed that the use of CE-FDTD algorithm does not bring significant improvement in any of the observed cases. Also, the increase of the accuracy order above four is not necessary for the observed case. The biggest contribution to the accuracy of the obtained results is provided by increasing the accuracy order of the finite differences in space from two to four.

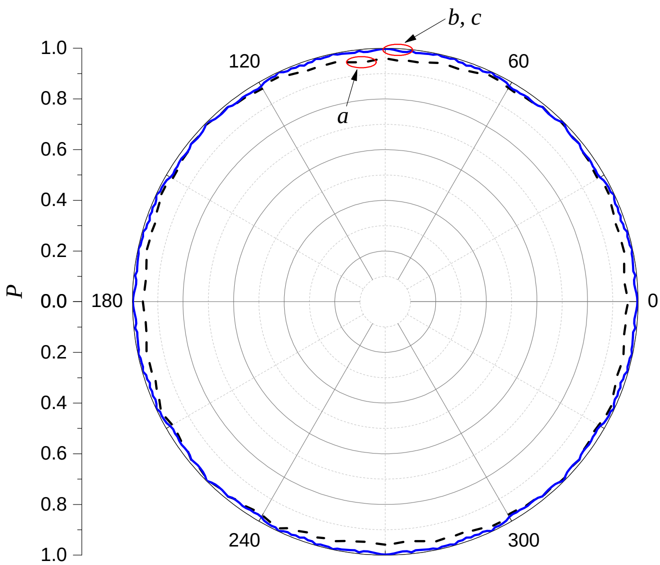

Fig. 5 – *Radiation pattern for: a – FDTD(2,2), b – FDTD(2,4), c – FDTD(2,6).*

6 Conclusions

In this paper an approach to efficient computation of radiation pattern in FDTD simulation environment is presented. A necessary large distance from the radiating object is achieved by multigrid space discretization with unilaterally connected subdomains. In order to combat the numerical error due to numerical dispersion more general CE-FDTD formulation and high order accuracy in space FDTD schemes are employed.

It is shown that the use of CE-FDTD algorithm does not bring significant improvement in any of the observed cases. The use of multigrid algorithm with two subdomains reduced the required computational power and memory consumption significantly and it does not produce significant undesireable reflections or disturbance of any other kind. The biggest contribution to the accuracy of the obtained results is provided by increasing the accuracy order of the finite differences in space from two to four. Further increase of the accuracy order does not bring significant improvements, but increase the complexity of the algorithm.

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