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The Numerical Method of Inverse Laplace Transform for Calculation of Overvoltages in Power Transformers and Test Results

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Abstract: A methodology for calculation of overvoltages in transformer windings, based on a numerical method of inverse Laplace transform, is presented. Mathematical model of transformer windings is described by partial differential equations corresponding to distributed parameters electrical circuits. The procedure of calculating overvoltages is applied to windings having either isolated neutral point, or grounded neutral point, or neutral point grounded through impedance. A comparative analysis of the calculation results obtained by the proposed numerical method and by analytical method of calculation of overvoltages in transformer windings is presented. The results computed by the proposed method and measured voltage distributions, when a voltage surge is applied to a three-phase 30 kVA power transformer, are compared.

Keywords: Numerical methods, Inverse Laplace transform, Transformer winding, Overvoltages, Transients.

1 Introduction

Transformer is one of the most common components in the power systems, but also the component that is extremely difficult to model accurately. Modeling the transformer windings is more complex than modeling the transmission line due to electrostatic and electromagnetic couplings among the winding elements. At impulse (or high frequency) voltages, a winding behaves as a complex network consisting of capacitances, resistances, self and mutual inductances [1 – 6]. There are two approaches to modeling the transformer at high frequencies. The first approach is to develop an equivalent circuit model having distributed parameters. By solving partial differential equations which describe the network having distributed parameters, the voltage distribution along the transformer windings is obtained. Analytical solution of the transients in the transformer windings requires some simplification because of complexity of the mathematical analysis. Another approach to solving the transient processes in the windings is dividing a winding in a finite number of elements and solving

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the relevant equations by numerical methods [4 – 6]. This approach allows modeling of the unequal distributions of resistances, inductances and capacitance along different types of windings.

The most commonly used procedure for solving partial differential equations describing the equivalent circuit having distributed parameters is to apply the Laplace transform. In solving partial differential equations by using the Laplace transform, the problem of finding analytically inverse Laplace transform appears. Therefore, numerical algorithms for finding the inverse Laplace transform have been developed [7 – 9].

This paper presents a methodology for calculation of overvoltages in the transformer windings, which is based on a numerical method of inverse Laplace transform [10]. A comparison between the numerically calculated and measured step responses has been made on a three-phase 30 kVA power transformer.

2 Numerical Method for the Inverse Laplace Transform

The algorithm for solving partial differential equations by using the Laplace transform starts from the set of equations in terms of time variable (with appropriate initial and boundary conditions) to which the Laplace transform is applied. The obtained ordinary differential equations are solved taking into account the boundary conditions, and then the inverse Laplace transform is applied.

Laplace transform $U(x, s)$ of the function $u=u(x, t)$, where $t\geq 0$ is the time variable, is:

$$U(x,s)=\mathcal{L}(u(x,t))=\int_0^{\infty}\mathrm{e}^{-st}u(x,t)\,\mathrm{d}t\,. \tag{1}$$

For the purpose of solving differential equations, the Laplace transform of the derivative of function $u=u(x, t)$ with respect to x and t variables is also used:

$$\mathcal{L}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}U(x,s)\,, \tag{2}$$

$$\mathcal{L}\left(\frac{\partial u}{\partial t}\right)=s\mathcal{L}(u(x,t))-u(x,0)=sU(x,s)-u(x,0)\,. \tag{3}$$

The inverse Laplace transform can be performed by using numerical algorithms for a range of values of the variable x within the desired interval. All numerical algorithms are based on the following relations [7]:

$$f(t)\approx f_n(t)\equiv\frac{1}{t}\sum_{k=0}^{n}\omega_k\hat{f}\left(\frac{\alpha_k}{t}\right),\quad 0<t<\infty\,, \tag{4}$$

where ω_k and α_k are complex parameters which depend on n, but do not depend on $\hat{f}$ or time t. Depending on the choice of ω_k and α_k, several algorithms are developed. One of the most used algorithms is Talbot's algorithm [8 – 9] which is defined by the following relations:

$$f_b(t,M) = \frac{2}{5t}\sum_{k=0}^{M-1} \mathrm{Re}\left(\gamma_k \hat{f}\left(\frac{\delta_k}{t}\right)\right), \tag{5}$$

$$\delta_k = \frac{2k\pi}{5}\left(\cot\left(k\pi/M\right) + i\right), \quad 0 < k < M\,, \tag{6}$$

$$\gamma_0 = e^{\delta_0}\ /\ 2\,, \tag{7}$$

$$\gamma_k = \left[1 + \mathrm{i}\left(k\pi/M\right)\left(1 + \left[\cot\left(k\pi/\pi\right)\right]^2\right) - i\cot\left(k\pi/M\right)\right]\mathrm{e}^{\delta_k}, \quad 0 < k < M, \tag{8}$$

3 Voltage Distribution Along a Transformer Winding

When a transformer is subjected to an impulse overvoltage, the transformer winding can be modeled by an equivalent scheme having uniformly distributed parameters, like the one shown in Fig. 1. The capacitances C with respect to the iron core and with respect to the transformer housing, inter-coil turn-to-turn capacitances K, self-inductances L of the winding coils, copper loss resistances r, and dielectric loss series and ground conductances g and G, respectively, are expressed per unit length of the winding. The voltage with respect to the ground at a distance x from the end of the windings is marked by u. Currents in the series branches having parameters K, L and g are marked by iK, iL and ig, respectively, and the currents in the parallel branches having parameters C and G are marked with iC and iG, respectively.

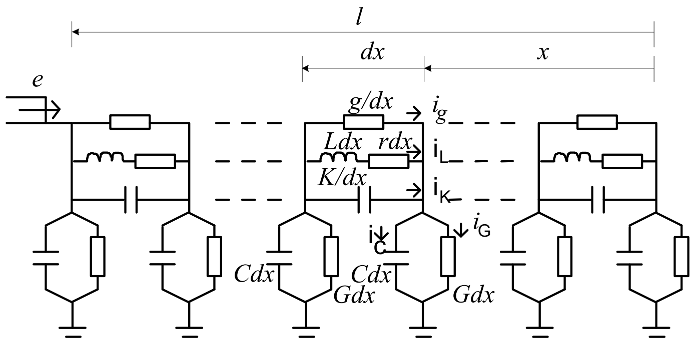

Fig. 1 – *The equivalent circuit of transformer winding.*

Analytical methods for the calculation of transients in the transformer windings are based on the distributed parameters equivalent circuit and differential equations which describe the distributed parameters network. Solving these equations means determining the initial voltage distribution, final voltage distribution, and voltage distribution in the transition period. The analytical solution is obtained as a product of oscillations in time and space which describe the process using standing or traveling waves. The analytical calculation methods based on standing waves take into account differently the impact of mutual inductance among the turns of the windings. The transition process in a winding is represented by the sum of an infinite series of harmonics, which oscillate in time and space, whose frequency and amplitude depend on the winding parameters and grounding impedance of the neutral point of the windings (impedance to ground at the end of the windings).

Basic equations of the circuit of Fig. 1 are:

$$i_K = K\frac{\partial^2 u}{\partial x\,\partial t}, \tag{9}$$

$$i_g = g\frac{\partial u}{\partial x}, \tag{10}$$

$$i_C + i_G = C\frac{\partial u}{\partial t} + Gu = \frac{\partial i_K}{\partial x} + \frac{\partial i_L}{\partial x} + \frac{\partial i_g}{\partial x}, \tag{11}$$

$$\frac{\partial u}{\partial x} = r\,i_L + L\frac{\partial i_L}{\partial t}. \tag{12}$$

From (1) – (4) the following partial differential equation is obtained:

$$LK\frac{\partial^4 u}{\partial x^2\partial t^2} + (rK + Lg)\frac{\partial^3 u}{\partial x^2\partial t} + (rg+1)\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + (rC + LG)\frac{\partial u}{\partial t} + rgu. \tag{13}$$

By using the Laplace transform with respect to all initial condition equal 0, (13) becomes:

$$(LKs^2 + (rK + Lg)s + rg + 1)\,\frac{\partial^2 U}{\partial x^2} = (LCs^2 + (rC + LG)s + rg)U\,. \tag{14}$$

The solution of the previous equation is of the form:

$$U(x,s) = A\,\mathrm{e}^{\sigma x} + B\,\mathrm{e}^{-\sigma x}, \tag{15}$$

where:

$$\sigma = \sigma(s) = \sqrt{\frac{LCs^2 + (rC + LG)s + rG}{LKs^2 + (rK + Lg)s + rg + 1}}\,. \tag{16}$$

Current through the winding is:

$$I(x,s) = \frac{1}{Z_s(s)}\frac{\partial U(x,s)}{\partial x} = \frac{1}{Z_s(s)}\left(A\sigma\,\mathrm{e}^{\sigma x} - B\sigma\,\mathrm{e}^{-\sigma x}\right), \tag{17}$$

where $Z_s(s)$ is a series impedance:

$$Z_s(s) = (r + Ls) / (KLs^2 + (gL + rK)s + gr + 1) . \quad (18)$$

If the neutral point of the windings is grounded through impedance $Z(s)$, the boundary conditions are:

$$U(l,s) = E(s) = A\mathrm{e}^{\sigma l} + B\mathrm{e}^{-\sigma l} , \quad (19)$$

$$U(0,s) = A + B = Z(s)I(0,s) = Z(s)(A\sigma - B\sigma) / Z_s(s) , \quad (20)$$

where $E(s)$ is the Laplace transform of the impulse overvoltage wave $e(t)$ with a steep front edge incoming the winding.

By calculating constants A and B and substituting them in (15):

$$U(x,s) = \frac{Z(s)\sigma\cosh(\sigma x) + Z_s(s)\sinh(\sigma x)}{Z(s)\sigma\cosh(\sigma l) + Z_s(s)\sinh(\sigma l)} E(s) . \quad (21)$$

The initial voltage distribution is determined by the capacitances and it is obtained for $s \to \infty$ according to the first boundary theorem of the Laplace transform of (21):

$$u_p(x) = \lim_{s\to\infty} sU(x,s) = \frac{\alpha\cosh(\alpha x) + \sinh(\alpha x)\lim\limits_{s\to\infty}\dfrac{Z_s(s)}{Z(s)}}{\alpha\cosh(\alpha l) + \sinh(\alpha l)\lim\limits_{s\to\infty}\dfrac{Z_s(s)}{Z(s)}} \lim_{s\to\infty} sE(s) , \quad (22)$$

where:

$$\alpha = \lim_{s\to\infty} \sigma = \sqrt{C / K} . \quad (23)$$

The final voltage distribution is determined by the resistances and it is obtained for $s \to 0$ according to the second boundary theorem of the Laplace transform of (21):

$$u_k(x) = \lim_{s\to 0} sU(x,s) = \frac{Z(0)\beta\cosh(\beta x) + Z_s(0)\sinh(\beta x)}{Z(0)\beta\cosh(\beta l) + Z_s(0)\sinh(\beta l)} \lim_{s\to 0} sE(s) , \quad (24)$$

where:

$$\beta = \lim_{s\to 0} \sigma = \sqrt{rG / (rg + 1)} . \quad (25)$$

For a direct grounded neutral point of the windings, it is $Z(s)=0$, so that (21) becomes:

$$U(x,s) = \frac{\sinh(\sigma x)}{\sinh(\sigma l)} E(s) . \quad (26)$$

For an isolated neutral point of the windings, $Z(s) \to \infty$, so that (21) becomes:

$$U(x,s)=\frac{\cosh(\sigma x)}{\cosh(\sigma l)}E(s)\,. \qquad (27)$$

By representing (26) and (27) as fractions of two symbolic functions and by applying Heaviside's expansion theorem, expressions for the voltage distribution along the transformer winding for the grounded and isolated neutral point of the windings are obtained:

$$u(x,t)=E\frac{\sinh(\beta x)}{\sinh(\beta l)}+\sum_k A_k\,e^{-\gamma_k t}\sin\left(\frac{k\pi x}{l}\right)\cos(\omega_k t)\,, \qquad (28)$$

$$u(x,t)=E\frac{\cosh(\beta x)}{\cosh(\beta l)}+\sum_k C_k e^{-\gamma_k t}\cos\left(\frac{k\pi x}{2l}\right)\cos(\omega_k\, t)\,, \qquad (29)$$

where:

$$A_k=\frac{2k\pi\left(\alpha^2-\beta^2\right)l^2\cos(k\pi)}{\left(\alpha^2 l^2+k^2\pi^2\right)\left(\beta^2 l^2+k^2\pi^2\right)}\,, \qquad (30)$$

$$C_k=-\frac{16k\pi\left(\alpha^2-\beta^2\right)l^2\sin\left(\frac{k\pi}{2}\right)}{\left(4\alpha^2 l^2+k^2\pi^2\right)\left(4\beta^2 l^2+k^2\pi^2\right)}E\,, \qquad (31)$$

$$\gamma_k=\frac{a_k^2(rK+Lg)+LG+rC}{2L(a_k^2K+C)}\,, \qquad (32)$$

$$\omega_k=\frac{\sqrt{\begin{array}{l}(2KL(rg+2)-g^2L^2-K^2r^2)a_k^4+\\ \qquad +(2Lr(GK+Cg)+4LC-2KCr^2-2gGL^2)a_k^4-(rC-LC)^2\end{array}}}{2L(a_k^2K+C)}\,. \qquad (33)$$

For a grounded neutral point, the spatial frequency of oscillations is:

$$a_k=k\pi/l\,. \qquad (34)$$

For an isolated neutral point, the spatial frequency of oscillations is:

$$a_k=k\pi/(2l)\,. \qquad (35)$$

The spatial frequencies of oscillations for the grounded and isolated winding end are obtained from the expression for inverse Laplace transform:

$$u(x,t)=\sum_{\substack{\text{in all poles of}\\ \text{function } U(s,t)}}\mathrm{Res}\left\{U(s,t)e^{st}\right\}. \qquad (36)$$

Voltage distribution along the transformer windings can be determined by the numerical procedure by applying numerical methods of inverse Laplace transform of (21).

4 Results of the Calculation of Voltage Distribution along a Transformer Winding

For the primary winding of a distribution transformer of rated power 30 kVA and rated voltages 10/0.4 kV/kV, the specified winding parameters are [5]:

$C = 1.28 \cdot 10^{-9}$ F/m, $K = 3.08 \cdot 10^{-12}$ Fm, $L = 1.11$ H/m,

$r = 200\ \Omega$/m, $G = 3.37 \cdot 10^{-6}$ S/m, $g = 1.9 \cdot 10^{-6}$ Sm.

Fig. 2 shows the initial and final voltage distributions along the primary winding with a grounded neutral point, obtained from (22), (24), and (26). Fig. 3 shows the initial and final voltage distributions along the primary winding with an isolated neutral point, obtained from (22), (24) and (27). Fig. 4 shows the voltage distribution in the transition period along the winding with a grounded neutral point, obtained according to the analytical and numerical methods. Fig. 5 shows the voltage distribution in the transition period along the winding with an isolated neutral point, obtained according to the analytical and numerical methods. Fig. 6 shows the voltage distribution in the transition period along the winding with a neutral point grounded through the inductance of 1 H, obtained by applying the numerical method. The same distribution as a function of time and winding length is shown in Fig. 7.

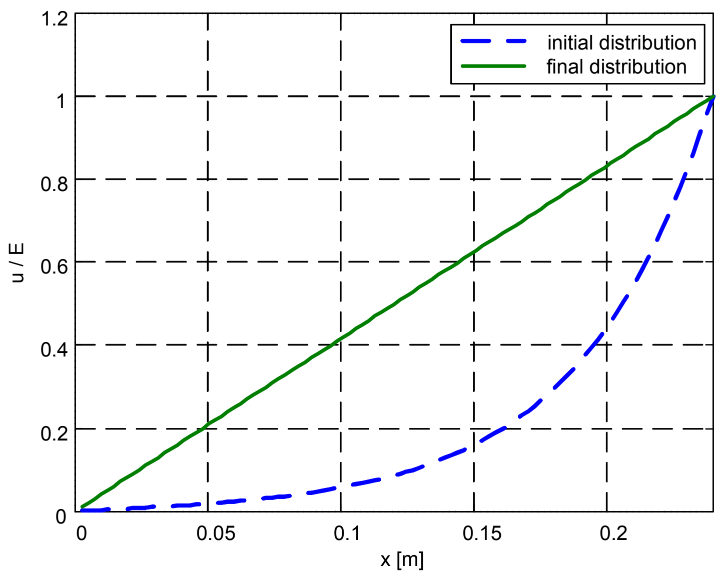

Fig. 2 – *The initial and final voltage distributions along the transformer winding with a direct grounded neutral point.*

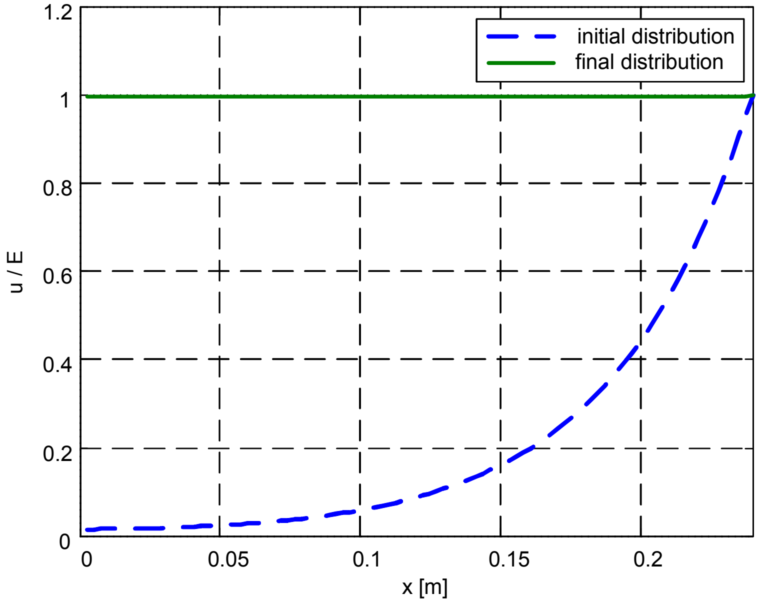

Fig. 3 – *The initial and final voltage distributions along the winding with an isolated neutral point.*

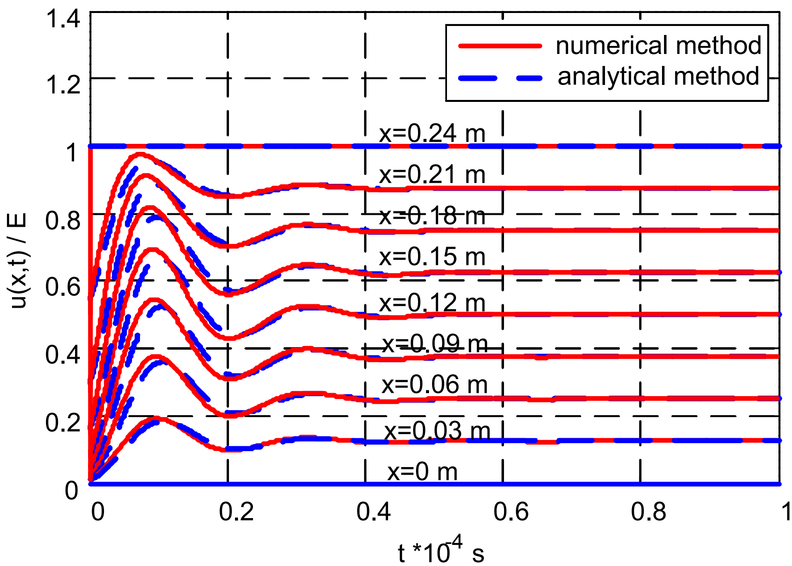

Fig. 4 – *The voltage distribution in the transition period along the winding with a grounded neutral point, obtained by the analytical and numerical methods.*

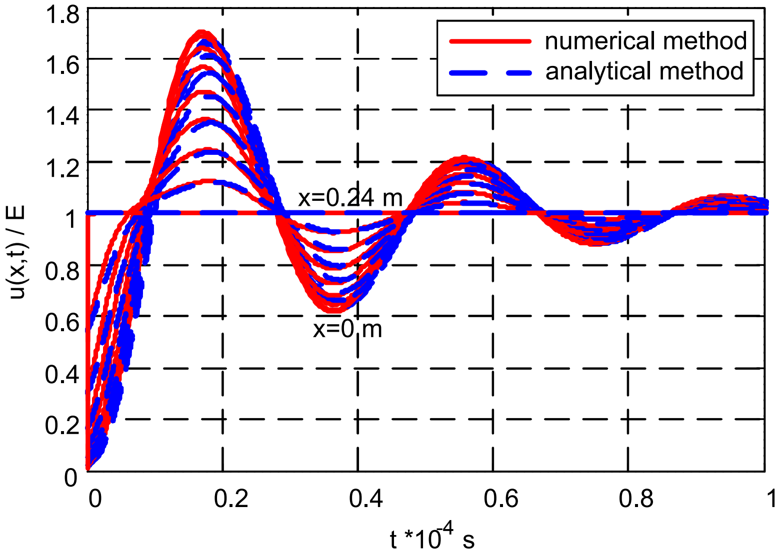

Fig. 5 – *The voltage distribution in the transition period along the windings with an isolated neutral point, obtained by the analytical and numerical methods.*

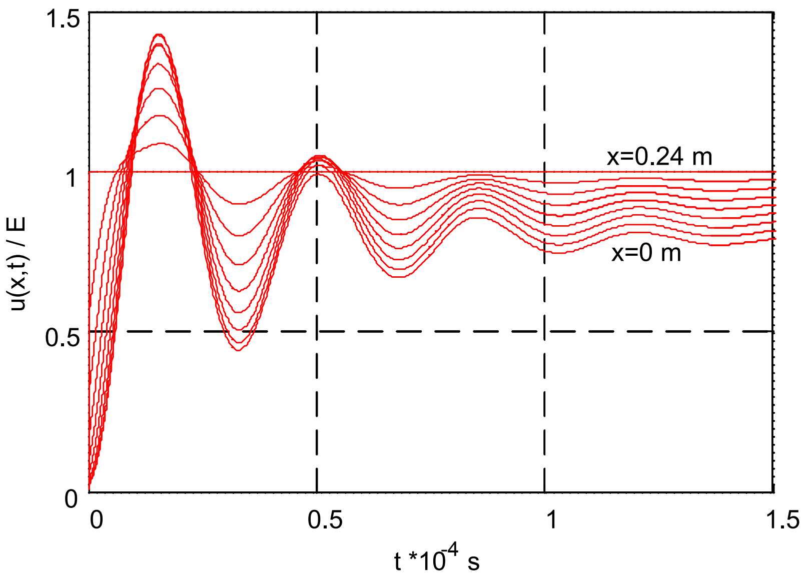

Fig. 6 – *The voltage distribution in the transition period along the winding with a neutral point grounded trough inductance, obtained by the numerical method.*

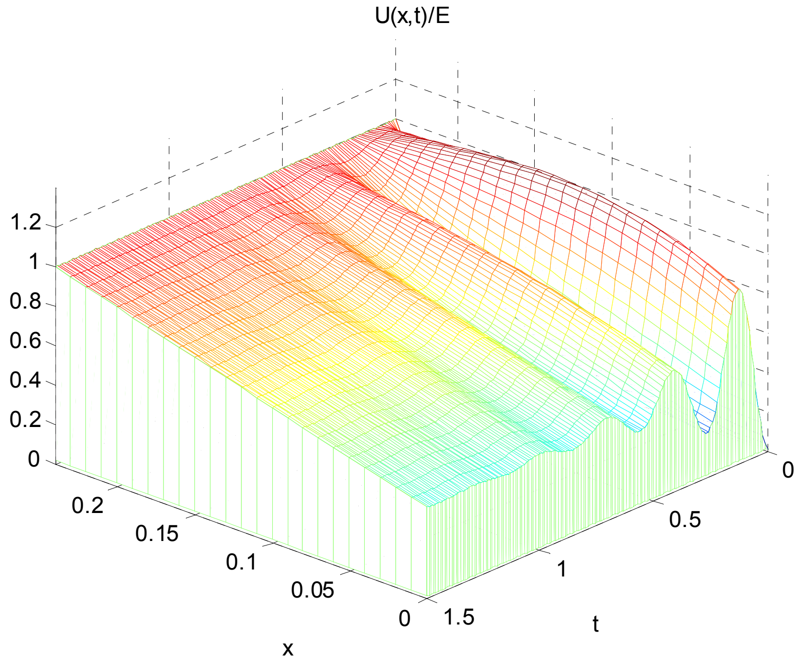

Fig. 7 – *The voltage distribution as a function of time t* [s] *and winding length x* [m] *for the winding with a neutral point grounded trough inductance, obtained by the numerical method.*

Based on the shown voltage distributions, it can be concluded that the voltages along the winding practically do not exceed amplitude of the incoming voltage surge at the winding beginning, in the case of a directly grounded neutral point of the windings. In the case of an isolated neutral point of the windings, at the end of the winding (at the neutral point) overvoltage of nearly double the amplitude of the incident wave, at the winding beginning, occurs. Therefore, turns of the winding close to the winding end are highly stressed, as well as the neutral point of the windings and in this case the insulation should be increased if there is no surge arrester to limit the voltage. Dangerous voltages in the turns of the winding close to the winding end (neutral point) can occur even when the neutral point of the winding is grounded through impedance.

Analytical calculation of the transients in the transformer winding is applied to a winding with directly grounded end and isolated neutral point, with some approximations due to complexity of the mathematical analysis. Transients in the transformer winding with respect to the mutual inductance between the turns and impedance connected at the neutral point of the winding are too complicated to solve by analytical methods. By using numerical inverse

Laplace transform, the voltage distribution along the transformer winding can be easily calculated when the neutral point of the windings is directly grounded, isolated or grounded through impedance. Modeling the mutual inductance between the turns inside the winding and solving the corresponding differential equations by using numerical Laplace transform method will be the subject of further research in this area.

5 Measurement Results and Comparison Between the Calculated and Measured Voltages

The performance of the proposed method is tested experimentally on a three-phase 10/0.4 kV/kV, 30 kVA power transformer having disc-coil windings. The repetitive surge generator tests have been made at high voltage windings. For this purpose, the transformer tank and oil isolation have been removed and the winding taps have been made. The voltage response along the windings has been measured by using conductively coupled probes.

In Fig. 8 the measurement method for the initial voltage distribution is shown. The measurement method for the final voltage distribution and voltage distribution throughout the transition period is shown in Fig. 9.

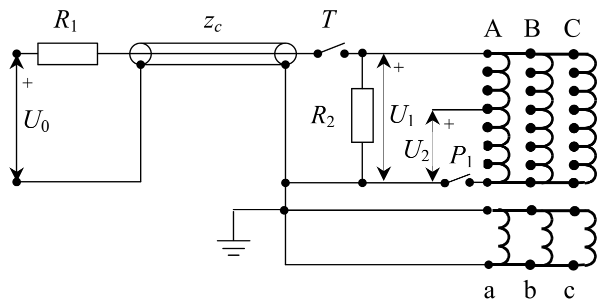

Fig. 8 – *The measurement method for the initial voltage distribution.*

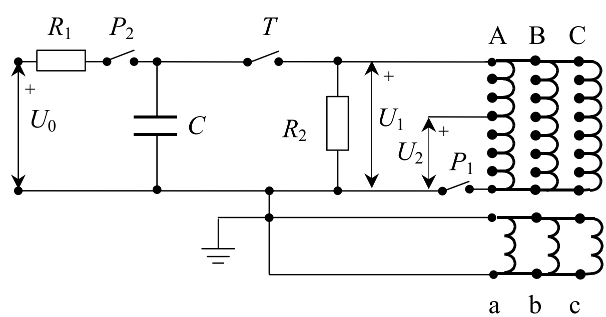

Fig. 9 – *The measurement method for the final voltage distribution and voltage distribution throughout the transition period.*

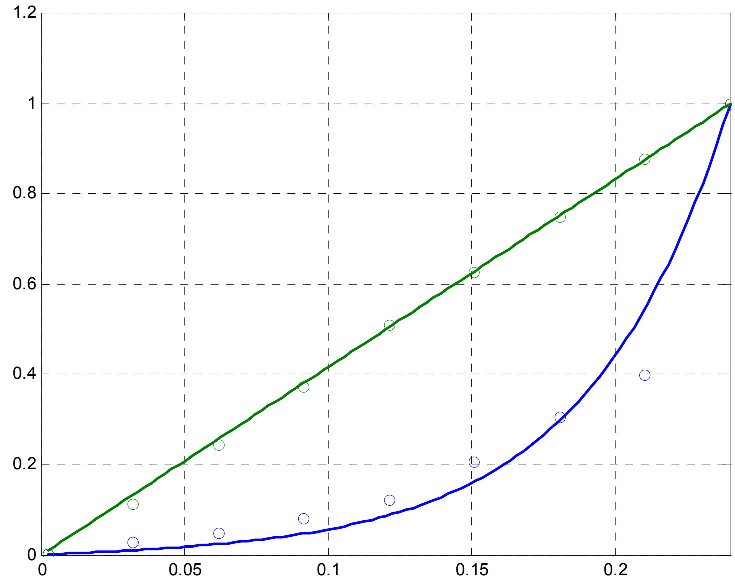

Fig. 10 – *The measured and calculated initial and final voltage distributions when the neutral point is directly grounded.*

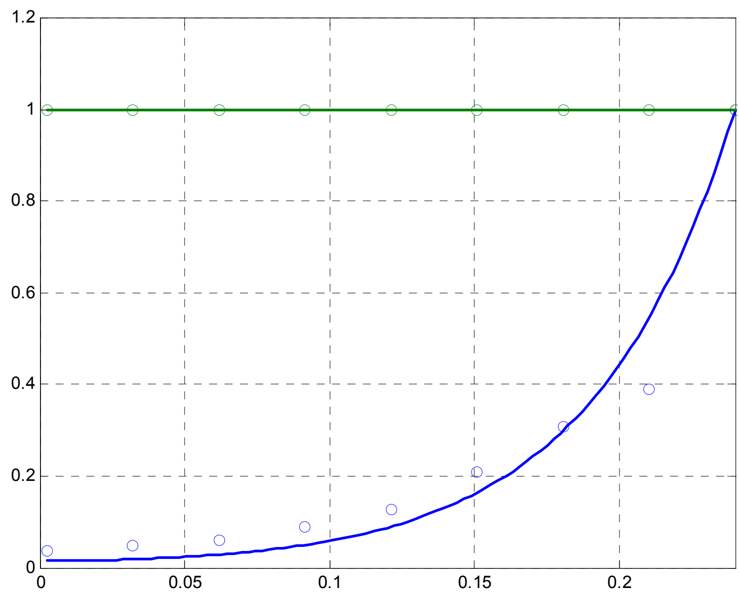

Fig. 11 – *The comparison between the measured and calculated initial and final voltage distributions when the neutral point is isolated.*

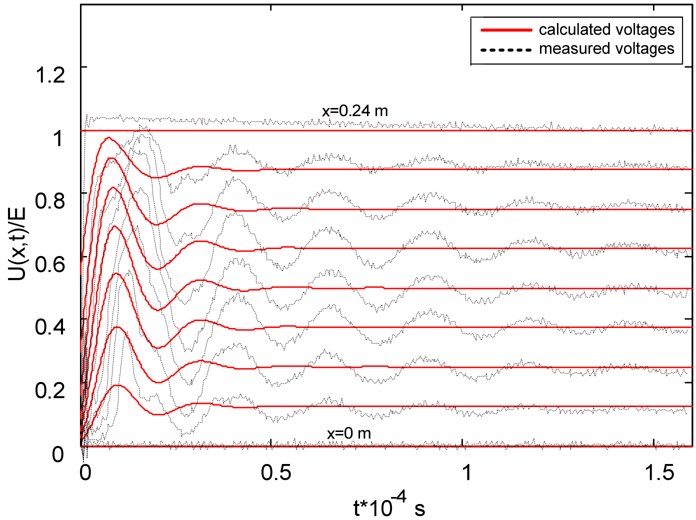

Fig. 12 – *The measured and calculated voltages throughout the transition period when the neutral point is directly grounded.*

The measured and calculated initial and final voltage distributions are shown in Fig. 10 when the neutral point is isolated and in Fig. 11 when the neutral point is grounded. The measured and calculated voltages along the winding with direct grounded neutral point during the transition period are shown in Fig. 12.

The calculated initial and final voltage distribution along the transformer windings showed good agreement with the measured values. The average difference between the calculated and measured initial voltage distributions is 2.96% (with a maximum of 5.69%) when the neutral point is isolated and 3.38% (with a maximum of 14.81%) when the neutral point is directly grounded. The average difference between the calculated and measured final voltage distributions, when the neutral is directly grounded, is 0.72% (with a maximum of 2.03%). The difference between the calculated and measured final voltage distributions in the winding with isolated neutral point is practically zero, since the steady-state voltages along the winding are equal to the amplitude of the incoming voltage surge at the winding's beginning. The average difference between the calculated and measured voltage distributions throughout the transition period, when the neutral point is grounded, is 10.40% (with a maximum of 18.24%). Obviously, a significant difference between the frequencies of the calculated and measured voltages throughout the transition period is obtained since mutual inductance between parts of the winding has been neglected. Also, the difficulties in determining the dielectric loss due to the series and ground conductances *g* and *G* in the model have caused too excessive damping of the voltages oscillations.

6 Conclusion

Analytical solution of transients in the transformer winding requires some approximations due to the complexity of the mathematical analysis. A more convenient approach to resolve the transients in a transformer winding is based on solving the relevant equations by numerical methods. This paper presents a comparative analysis of the results of the calculation of voltage distributions in a transformer winding by using the analytical and numerical Laplace transform methods. The analyses have justified the necessity of development and application of numerical methods for the calculation of transients in transformer windings. A comparison between the numerically calculated and measured step responses has been made on a three-phase 30 kVA power transformer. The tests made on this transformer showed good agreement between the calculated and measured results for the initial and final distributions and a meaningful deviation between the distributions throughout the transition period because of parameter imprecision in the transformer winding model.

7 Acknowledgement

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