SERBIAN JOURNAL OF ELECTRICAL ENGINEERING
Vol. 8, No. 1, February 2011, 9-15

UDK: 517.15

Some Consequence Relations on Propositional Formulas*

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Abstract: Consequence relations on propositional formulas are binary relations on propositional formulas that represent certain types of entailment – formal or semi-formal derivation of conclusion from a certain set of premises. Some of well known examples are classical implication (standard logical entailment), preference relations (i.e. relations that satisfy Reflexivity, Left logical equivalence, Right weakening, And, Or and Cautious monotonicity) rational relations (i.e. preference relations that also satisfy rational monotonicity), consequence relations (prime examples are qualitative possibilities and necessities) etc. More than two decades various consequence relations are used in automated decision making, product control, risk assessment and so on. The aim of this paper is to give a short overview of the most prominent examples of consequence relations.

Keywords: Propositional formulas, Classical implication, Preference relations.

1 Introduction

Consequence relations on formulas are binary relations on formulas that represent certain reasoning process. Some problems such as risk assessment, classification of persons/objects according to predefined criteria, finding acceptable bargain in a negotiation with contradictory demands etc. are especially suited for nonclassical reasoning.
Prime examples such reasoning processes are default reasoning, counterfactual reasoning, probabilistic and fuzzy reasoning, possibilistic reasoning etc.

The aim of this paper is to give a short overview of some important consequence relations. There are many types of these relations that are relevant for the artificial intelligence and automated reasoning. Our choice is to present two kinds of those relations: confidence relations and preference relations.

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*Award for the best paper presented in Section *Artificial Intelligence*, at Conference ETRAN 2010, June 7-11, Donji Milanovac, Serbia.

Concerning confidence relations, we will give some basic properties of comparative probabilities (historically the first example of confidence relations), and what we believe that are the prime example of confidence relations - qualitative possibilities and necessities. This belief is also reflected in the list of references – most of them are connected to the possibility theory.

Concerning preference relations, we give the standard definition of a preference relation, and the well known extension - the notion of a rational relation. A landmark in this field is the research of Kraus, Lehmann and Magidor published in the early nineties. Characterization theorems of Kraus, Lehmann and Magidor are listed here as well.

Finally, we have not give the precise account of the authors for mentioned contribution in terms of pointers to the list of references. However, we have try to give a name for each contributor, so it should not be difficult to divine the right reference from the given list.

2 Preliminaries

The purpose of this section is to fix notation that will be used throughout the rest of the paper.

Propositional letters (propositional variables) will be denoted by p and q, indexed or primed if necessary. The set of all propositional letters will be denoted by *Var*. Classical propositional formulas built over the set of propositional letters *Var* will be denoted by α and β, indexed or primed if necessary. The set of all classical propositional formulas will be denoted by For_C.

Variables (formal names) for binary relations on For_C are R and S, indexed or primed if necessary. We will use the standard infix notation, i.e. $\alpha\ R\ \beta$ reads "α is in relation R with β". Since the context is always clear, the same symbols will be used to denote actual binary relations on For_C.

By **B** we will denote the Lindenbaum's propositional algebra over the set of propositional letters *Var*. Recall that **B** is the quotient set $For_C/\sim$, where "~" is congruence on propositional formulas defined by:

$$\alpha \sim \beta \text{ iff } \alpha \text{ is equivalent with } \beta.$$

Corresponding equivalence classes will be denoted by $\alpha^{\mathbf{B}}$. Boolean operations (meet), + (join), ^{c} (complement) and constants **1** and **0** are defined as follows:

1. $\alpha^{\mathbf{B}} \cdot \beta^{\mathbf{B}} = (\alpha \otimes \beta)^{\mathbf{B}}$;
2. $\alpha^{\mathbf{B}} + \beta^{\mathbf{B}} = (\alpha \oplus \beta)^{\mathbf{B}}$;
3. $(\alpha^{\mathbf{B}})^c = (\neg\, \alpha)^{\mathbf{B}}$;
4. $\mathbf{1} = (p \oplus \neg\, p)^{\mathbf{B}}$;
5. $\mathbf{0} = (p \otimes \neg\, p)^{\mathbf{B}}$.

If R is a binary relation of For_C, then $R^{\mathbf{B}}$ is its image under the mapping $(\alpha, \beta) \rightarrow (\alpha^{\mathbf{B}}, \beta^{\mathbf{B}})$.

In order to formalize reasoning about consequence relations, we will extend the classical propositional language with formulas of the form $(\alpha\ R\ \beta)$ as the basic consequence formulas, and define consequence formulas as Boolean combinations of basic consequence formulas.

Consequence formulas will be denoted by φ and ψ, indexed or primed if necessary. The set of all consequence formulas will be denoted by *For*.

In order to simplify notation, for the actual binary relation R on For_C, $\alpha <_R \beta$ means that $\alpha\ R\ \beta$ and not $\beta\ R\ \alpha$. Syntactically, $\alpha <_R \beta$ is an abbreviation of the formula:

$$(\alpha\ R\ \beta) \otimes \neg\ (\beta\ R\ \alpha),$$

where in this context R is a name (syntactical symbol) for a binary relation on For_C.

Formally, a hyperreal number is an element of some nonarchimedean ω_1-saturated elementary extension of the ordered field of reals. In practice, we may think of hyperreal numbers as ordinary real numbers with addition of infinitesimals and infinitely large numbers. A positive infinitesimal is a positive hyperreal number that is lesser than all positive ordinary real numbers. Existence of such objects can be easily established using the compactness theorem for the first order predicate logic.

3 Confidence Relations

A binary relation R on For_C is said to be a confidence relation if it has the following properties:

1. Reflexivity: $\alpha\ R\ \alpha$;
2. Linearity: $\alpha\ R\ \beta$ or $\beta\ R\ \alpha$;
3. Transitivity: if $\alpha_1\ R\ \alpha_2$ and $\alpha_2\ R\ \alpha_3$, then $\alpha_1\ R\ \alpha_3$;
4. Coherence with deduction: $\alpha\ R\ \beta$ whenever α implies β;
5. Nontriviality: if α is a contradiction and β is a tautology, then $\alpha <_R \beta$;
6. Weak stability: if $(\alpha_1 \oplus \alpha_2) \otimes \alpha_3$ is a contradiction, then $\alpha_1 <_R \alpha_2$ implies that $(\alpha_1 \oplus \alpha_3)\ R\ (\alpha_2 \oplus \alpha_3)$.

The first example of a confidence relation is a comparative probability relation, introduced by Bruno De Finetti in 1937, and exploited later by Savage in the early seventies in decision theory. A comparative probability is a binary relation on propositional formulas that satisfies reflexivity, linearity, transitivity, nontriviality and the following three properties:

1. Consistency: if α is a contradiction, then $\alpha\ R\ \beta$ for any formula β;
2. Equivalence: if $\alpha\ R\ \beta$, then $\alpha'\ R\ \beta'$, whenever α is equivalent to α' and β is equivalent to β';
3. Preadditivity: if $(\alpha_1 \oplus \alpha_2) \otimes \alpha_3$ is a contradiction, then $\alpha_1\ R\ \alpha_2$ iff $(\alpha_1 \oplus \alpha_3)\ R\ (\alpha_2 \oplus \alpha_3)$.

It is easy to see that any comparative probability is a confidence relation. Indeed, preadditivity is a stronger condition than weak stability, so it only remains to verify coherence with deduction. Indeed, suppose that α implies β. Then, $\alpha \oplus \beta$ is equivalent to β, so $(\alpha \oplus \beta) \otimes \neg\,\beta$ is a contradiction. Thus, by preadditivity,

$$\alpha\ R\ \beta \text{ iff } (\alpha \oplus \neg\,\beta)\ R\ (\beta \oplus \neg\,\beta).$$

It remains to show that $\alpha'\ R\ \beta'$ whenever β' is a tautology. Suppose that α' is an arbitrary propositional formula and that α'' is any contradiction. Then, $(\neg\,\alpha' \oplus \alpha'') \otimes \alpha'$ is a contradiction. By preadditivity,

$$\alpha''\ R \neg\,\alpha' \text{ iff } (\alpha'' \oplus \alpha')\ R\ (\neg\,\alpha' \oplus \alpha').$$

By consistency and equivalence, $\alpha'\ R\ \beta'$, where β' is any tautology.

More prominent examples of confidence relations are qualitative possibility and qualitative necessity relations. Those relations were introduced by Lewis in 1973 and then independently defined by Dubois in 1986. The theory of possibility was extensively developed by Dubois, Prade and many others.

A qualitative possibility relation is any binary relation R on For_C that satisfies reflexivity, equivalence, linearity, transitivity, nontriviality and the following two properties:

1. If α is a contradiction, then α R β for all β;
2. Disjunctive stability: α_1 R α_2 implies that $(\alpha_1 \oplus \alpha_3)$ R $(\alpha_2 \oplus \alpha_3)$.

Dually, a qualitative necessity relation is any binary relation R on For_C that satisfies reflexivity, equivalence, linearity, transitivity, nontriviality and the following two dual properties:

1. If α is a tautology, then β R α for all β;
2. Conjunctive stability: α_1 R α_2 implies that $(\alpha_1 \otimes \alpha_3)$ R $(\alpha_2 \otimes \alpha_3)$.

It is easy to see that every qualitative possibility relation R generates unique qualitative necessity relation S_R by

$$\alpha\ S_R\ \beta \text{ iff} \neg\,\beta\ R \neg\,\alpha.$$

Moreover, any qualitative necessity relation S generates unique qualitative possibility relation R_S by

$$\alpha\ R_S\ \beta \text{ iff} \neg\,\beta\ S \neg\,\alpha.$$

Important common features of all confidence relations are corresponding distribution functions. Namely, a function f that maps classical propositional formulas in some linear ordering $(L,\leq)$ is a distribution function for a confidence relation R if

$$\alpha\ R\ \beta \text{ iff } f(\alpha) \leq f(\beta).$$

It is easy to show that any confidence relation has a proper class many distribution functions. Usually, the set of propositional letters is at most countably infinite, so we may restrict distribution functions to those which domain is the real unit interval [0,1].

For example, if f is a distribution function for a qualitative possibility relation R, then

$$f(\alpha \oplus \beta) = \max\ (f(\alpha), f(\beta)).$$

3 Preference Relations

A preference relation is any binary relation R on the set of classical propositional formulas with the following properties:

1. Reflexivity: α R α;
2. Left logical equivalence: if $\alpha\ R\ \beta$, then $\alpha'\ R\ \beta$ for all α' equivalent to α;
3. Right weakening: if $\alpha\ R\ \beta$, then $\alpha\ R\ \beta'$ whenever β implies β';
4. And: if $\alpha_1\ R\ \alpha_2$ and $\alpha_1\ R\ \alpha_3$, then $\alpha 1\ R\ (\alpha_2 \otimes \alpha_3)$;
5. Or: if $\alpha_1\ R\ \alpha_3$ and $\alpha_2\ R\ \alpha_3$, then $(\alpha_1 \oplus \alpha_2)\ R\ \alpha_3$.

Cautious monotonicity: if $\alpha_1\ R\ \alpha_3$ and $\alpha_1\ R\ \alpha_3$, then $(\alpha_1 \otimes \alpha_2)\ R\ \alpha_3$.

In 1990 Kraus, Lehmann and Magidor were give a full characterization of preference relations via preferential structures. A preferential structure is a triple $(S, <, l)$ where:

1. S is a nonempty set of states;
2. l: S $\rightarrow$ $P(W)$ is a label function. Here W is the set of all classical evaluations of propositional letters (worlds) and $P(W)$ is the powerset of W;
3. < is a strict partial ordering on S that satisfies the following smoothness condition: for all α, the set $[\alpha]$ of all states s such that $w(\alpha)=1$ for all w in $l(s)$ is smooth, i.e. either $[\alpha]$ is empty, or $[\alpha]$ is nonempty and each s in $[\alpha]$ is either minimal in $[\alpha]$, or it is above some minimal element in$[\alpha]$.

Let $M = (S, <, l)$ be a preferential structure. Then, a binary relation R_M defined by $\alpha\ R_M\ \beta$ iff either $[\alpha]$ is empty, or $[\alpha]$ is nonempty and $w(\beta) = 1$ for all minimal s in $[\alpha]$ and all w in $l(s)$.

Kraus, Lehmann and Magidor have proved that for each preference relation R exists preferential structure M such that $R = R_M$.

A rational relation is a preference relation R on formulas that additionally satisfies the following condition also known as rational monotonicity: if $\alpha_1 R \alpha_3$ and not $\alpha_1 R \neg \alpha_2$, then $(\alpha_1 \otimes \alpha_2) R \alpha_3$.

Lehmann and Magidor have provided a complete characterization of rational relations in terms of hyperreal-valued conditional probabilities. Namely, for each rational relation R exists hyperreal-valued finitely additive probability measure m_R such that $\alpha R \beta$ iff $1 - m_R(\alpha \mid \beta)$ is an infinitesimal. Recall that $a > 0$ is an infinitesimal iff $na < 1$ for every positive integer n.

4 Conclusion

In this short overview article we have presented several important types of consequence relations on propositional formulas that have been applied in automated reasoning for more than two decades. Consequence relations are directly connected to certain types of reasoning: beliefs, counterfactuals, default reasoning, probability reasoning etc. This paper represents the initial stage in the development of a formal system that will formally capture majority of known consequence relations.

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