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On the K and K_G Fading Channels

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Abstract: In this paper, the composite Rayleigh-Gamma and Nakagami-Gamma distributions are considered. Multipath short-term fading and long-term fading (shadowing) affect wireless channels. A composite fading model was proposed for the modeling of shadowed channels, which resulted in a closed form solution for the probability density function (pdf). These composite Rayleigh-Gamma and Nakagami-Gamma are well-known as the K-distribution and K_G-distribution, and are applied to cases where both micro- and macro-diversity schemes are implemented to mitigate short-term fading and shadowing, respectively. Thus, the composite pdf model offers significant improvement over approaches, which use lognormal pdf for shadowing. The results demonstrate the simplicity and usefulness of the composite pdf in the performance analyses of shadowed fading channels.

Keywords: Fading, Compound Gamma distribution, Nakagami-*m*, Rayleigh, RV, Shadowing.

1 Introduction

The composite Rayleigh-Gamma and Nakagami-Gamma distributions are mathematically tractable for the analytical evaluation, a bit error rate and an outage probability of communication systems. The composite Rayleigh–Gamma distribution is known as the K-distribution and the composite Nakagami-Gamma distribution is known as the K_G-distribution. The composite Rayleigh-Gamma distribution closely approximates Rayleigh–lognormal distribution and the composite Nakagami-Gamma distribution closely approximates Nakagami-lognormal distribution.

The composite Rayleigh-lognormal and Nakagami-lognormal distributions are not mathematically tractable for analytical evaluations. Multipath fading and

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shadow fading can be modeled by the composite Rayleigh-Gamma and Nakagami-Gamma distributions.

Using the K and K_G fading models we can obtain closed form and easy use of expressions for the average BER.

The main drawback of the Rayleigh-lognormal distribution is its complicated mathematical form. Multipath fading and shadow fading affect wireless channels and the composite Rayleigh- Gamma and Nakagami-Gamma are proposed for the modeling of shadowed fading channels. This model is applied to the cases where both micro- and macro-diversity are used to mitigate short-term fading and long-term fading.

Channels suffer from short-term fading and long-term fading

None of Rayleigh and Nakagami models leads to a closed form solution for the probability density function of the signal for noise at the output of receiver. Recently, composite Rayleigh– Gamma and a composite Nakagami-Gamma distribution are proposed to describe the shadowed fading channels.

This model assumes a Nakagami or Rayleigh density function for the envelope of the received signal and the power of the envelope has gamma distribution. This composite model can be used to analyze the performance of wireless systems in shadowed fading channels.

A multipath short-term fading is mitigated using micro-diversity techniques and it will not be sufficient to mitigate channel degradation when shadowing is present. Long-term fading or shadowing fading is mitigated by using macro diversity techniques.

This paper is organized as follows. In the Section 2 the probability density function and the moment generating function (MGF) of Rayleigh-Gamma distribution are determined; in the Section 3 the bivariate Rayleigh-Gamma distribution is considered; in the Section 4 Nakagami-Gamma distribution is analyzed, and in the Section 5 bivariate Nakagami-Gamma distribution is calculated. In the Section 6 the conclusion is given.

2 Statistics of Rayleigh-Gamma Random Variable

The conditional pdf of Rayleigh random variable is:

$$\Pr\left(r / y\right) = \frac{2r}{y} \mathrm{e}^{-\frac{r^2}{y}}, \quad r \ge 0, \tag{1}$$

where $y = \overline{r^2}$, if $\gamma = r^2$, $r = \sqrt{\gamma}$, and $\frac{\mathrm{d}r}{\mathrm{d}\gamma} = \frac{1}{2\sqrt{\gamma}}$. We can obtain the conditional pdf of squared Rayleigh random variable:

$$P\gamma\left(\gamma / y\right)=\left|\frac{\mathrm{d}r}{\mathrm{d}\gamma}\right|P\gamma\left(\sqrt{\gamma}/y\right)=\frac{1}{y}\mathrm{e}^{-\frac{\gamma}{y}}. \tag{2}$$

Averaging the expression (2) with respect to y, we obtain Rayleigh distribution in the term:

$$P\gamma(\gamma)=\int_0^\infty P\gamma\left(\gamma / y\right)Py\left(y\right)\mathrm{d}\,y. \tag{3}$$

The power of random variable γ has Gamma distribution:

$$Py(y)=\frac{1}{\Gamma(c)}y_0^{-c}y^{c-1}\,\mathrm{e}^{-\frac{y}{y_0}}. \tag{4}$$

In $Py(y)$, c is the order of the Gamma distribution and it is the measure of the shadowing present in the channel. By substituting (2) and (4) in (3) we can write:

$$P\gamma(\gamma)=\int_0^\infty \frac{1}{y}\mathrm{e}^{-\frac{\gamma}{y}}\frac{1}{\Gamma(c)}y_0^{-c}y^{c-1}\,\mathrm{e}^{-\frac{y}{y_0}}\,\mathrm{d}\,y=\frac{y_0^{-c}}{\Gamma(c)}\int_0^\infty y^{c-2}\,\mathrm{e}^{-\frac{\gamma}{y}-\frac{y}{y_0}}\,\mathrm{d}\,y. \tag{5}$$

Integral form of the modified Bessel function of second kind is:

$$\int_0^\infty x^{\upsilon-1}\,\mathrm{e}^{-\frac{\beta}{x}-\gamma x}\,\mathrm{d}\,x=2\left(\frac{\beta}{\gamma}\right)^{\frac{\upsilon}{2}}K_\upsilon\left(2\sqrt{\beta\gamma}\right). \tag{6}$$

By substituting (6) in (5), we obtain pdf of Rayleigh in the equation:

$$P\gamma(\gamma)=2\frac{y_0^{-c}}{\Gamma(c)}(\gamma y_0)^{\frac{1}{2}(c-1)}K_{c-1}\left(2\sqrt{\frac{\gamma}{y_0}}\right). \tag{7}$$

The cumulative distribution function of the Rayleigh is:

$$F\gamma(v)=\int_0^\gamma P\gamma(x)\,\mathrm{d}\,x. \tag{8}$$

Substituting (5) in (8) results in

$$F\gamma(\gamma)=\int_0^\gamma \frac{y_0^{-c}}{\Gamma(c)}\mathrm{d}\,x\int_0^\infty y^{c-2}\,\mathrm{e}^{-\frac{x}{y}-\frac{y}{y_0}}\,\mathrm{d}\,y. \tag{9}$$

Changing the order of integrations can be written as:

$$
\begin{aligned}
F\gamma(\gamma) &= \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} \mathrm{d}\,y \cdot y^{c-2}\,\mathrm{e}^{-\frac{y}{y_0}}\int_0^{\gamma} \mathrm{e}^{-\frac{x}{y}}\,\mathrm{d}\,x = \\
&= \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} \mathrm{d}\,y \cdot y^{c-2}\,\mathrm{e}^{-\frac{y}{y_0}}\,y\left[\mathrm{e}^{-\frac{x}{y}}\right]_0^{\gamma} = \\
&= \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} \mathrm{d}\,y \cdot y^{c-1}\,\mathrm{e}^{-\frac{y}{y_0}}\left(1-\mathrm{e}^{\frac{\gamma}{y}}\right) = \\
&= \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} y^{c-1}\,\mathrm{e}^{-\frac{y}{y_0}}\,\mathrm{d}\,y - \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} y^{c-1}\,\mathrm{e}^{-\frac{\gamma}{y}-\frac{y}{y_0}}\,\mathrm{d}\,y = \\
&= \frac{y_0^{-c}}{\Gamma(c)}\,y_0^{c}\Gamma(c) - \frac{y_0^{-c}}{\Gamma(c)}\,2(\gamma y_0)^{\frac{c}{2}} K_c\left(2\sqrt{\frac{\gamma}{y_0}}\right) = \\
&= 1-\frac{2}{\Gamma(c)}\left(\frac{\gamma}{y_0}\right)^{\frac{c}{2}} K_c\left(2\sqrt{\frac{\gamma}{y_0}}\right).
\end{aligned}
\tag{10}
$$

The moment generating function (MGF) of r is:

$$
M\gamma(s) = \overline{\mathrm{e}^{rs}} = \int_0^{\infty} \mathrm{e}^{rs}\,P\gamma(\gamma)\,\mathrm{d}\gamma\,. \tag{11}
$$

Substituting (5) in (11) and changing the order of integrations we can write:

$$
\begin{aligned}
M\gamma(s) &= \left[\int_0^{\infty} \mathrm{d}\gamma\,\mathrm{e}^{\gamma s}\,\frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} \mathrm{d}\,y \cdot y^{c-2}\,\mathrm{e}^{-\frac{\gamma}{y}-\frac{y}{y_0}}\,y\,\mathrm{e}^{-\frac{\gamma}{y}}\right] = \\
&= \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} \mathrm{d}\,y \cdot y^{c-2}\,\mathrm{e}^{-\frac{y}{y_0}}\int_0^{\infty} \mathrm{d}\gamma\,\mathrm{e}^{-\frac{\gamma}{y}}\,e^{\gamma s} = \\
&= \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} \mathrm{d}\,y \cdot y^{c-2}\,\mathrm{e}^{-\frac{y}{y_0}}\int_0^{\infty} \mathrm{d}\gamma\,\mathrm{e}^{-\gamma\left(\frac{1}{y}-s\right)} = \\
&= \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} \mathrm{d}\,y \cdot y^{c-2}\,\mathrm{e}^{-\frac{y}{y_0}}\,\frac{y}{1-ys} = \\
&= \frac{y_0^{-c}}{\Gamma(c)}\int_0^{\infty} y^{c-1}\left(1-ys\right)^{-1}\mathrm{e}^{-\frac{y}{y_0}}\,\mathrm{d}\,y.
\end{aligned}
\tag{13}
$$

Moment n-th order of γ is:

$$m_n = \overline{\gamma^n} = \int_0^\infty \gamma^n P\gamma(\gamma)\,\mathrm{d}\gamma\,. \tag{14}$$

Substituting (5) in (14) and changing the order of integrations the moment of n-th order of γ becomes:

$$\begin{aligned} m_n &= \int_0^\infty \gamma^n\,\mathrm{d}\gamma \frac{y_0^{-c}}{\Gamma(c)}\int_0^\infty \mathrm{d}y\cdot y^{c-2}\,\mathrm{e}^{-\frac{\gamma}{y}-\frac{y}{y_0}}\,y\,\mathrm{e}^{-\frac{\gamma}{y}} = \\ &= \frac{y_0^{-c}}{\Gamma(c)}\int_0^\infty \mathrm{d}y\cdot y^{c-2}\,\mathrm{e}^{-\frac{\gamma}{y}-\frac{y}{y_0}}\int_0^\infty \mathrm{d}\gamma\cdot\gamma^n\,\mathrm{e}^{-\frac{\gamma}{y}} = \\ &= \frac{y_0^{-c}}{\Gamma(c)}\int_0^\infty \mathrm{d}y\cdot y^{c-2}\,\mathrm{e}^{-\frac{y}{y_0}}\,y^{n+1}\Gamma(n+1) = \\ &= \frac{y_0^{-c}}{\Gamma(c)}\int_0^\infty \mathrm{d}y\cdot y^{c+n-1}\,\mathrm{e}^{-\frac{y}{y_0}} = \frac{\Gamma(c+n)}{\Gamma(c)}y_0^n. \end{aligned} \tag{15}$$

The random variables γ_1 and γ_2 are Rayleigh distributed:

$$\begin{aligned} P\gamma_1(\gamma_1) &= \frac{y_{01}^{-c_1}}{\Gamma(c_1)}\int_0^\infty y_1^{c_1-1}\,\mathrm{e}^{-\frac{\gamma_1}{y_1}-\frac{y_1}{y_{01}}}\,\mathrm{d}y_1, \\ P\gamma_2(\gamma_2) &= \frac{y_{02}^{-c_2}}{\Gamma(c_2)}\int_0^\infty y_2^{c_2-1}\,\mathrm{e}^{-\frac{\gamma_2}{y_2}-\frac{y_2}{y_{02}}}\,\mathrm{d}y_2. \end{aligned} \tag{16}$$

The pdf of sum of γ_1 and γ_2 is:

$$\begin{aligned} &P\gamma(\gamma) = \int_0^\gamma P\gamma_1(\gamma|\gamma_2)P\gamma_2(\gamma_2)\,\mathrm{d}\gamma_2 = \\ &= \int_0^\gamma \frac{y_{01}^{-c_1}}{\Gamma(c_1)}\frac{y_{02}^{-c_2}}{\Gamma(c_2)}\mathrm{d}\gamma_2\int_0^\infty \mathrm{d}y_1 y_1^{c_1-2}\,\mathrm{e}^{-\frac{\gamma-\gamma_2}{y_1}-\frac{y_1}{y_{01}}}\int_0^\infty \mathrm{d}y_2 y_2^{c_2-2}\,\mathrm{e}^{-\frac{\gamma_2}{y_2}-\frac{y_2}{y_{02}}}. \end{aligned} \tag{17}$$

Changing the order of integrations, we can write:

$$\begin{aligned} P\gamma(\gamma) &= \frac{y_{01}^{-c_1}}{\Gamma(c_1)}\frac{y_{02}^{-c_2}}{\Gamma(c_2)}\int_0^\infty \mathrm{d}y_1 y_1^{c_1-2}\,\mathrm{e}^{-\frac{\gamma}{y_1}-\frac{y_1}{y_{01}}}\int_0^\infty \mathrm{d}y_2 y_2^{c_2-2}\,\mathrm{e}^{-\frac{y_2}{y_{02}}}\int_0^\gamma \mathrm{d}y_2\,\mathrm{e}^{-\frac{\gamma_2}{y_1}-\frac{y_2}{y_{02}}} = \\ &= \frac{y_{01}^{-c_1}}{\Gamma(c_1)}\frac{y_{02}^{-c_2}}{\Gamma(c_2)}\int_0^\infty \mathrm{d}y_1 y_1^{c_1-2}\,\mathrm{e}^{-\frac{\gamma}{y_1}-\frac{y_1}{y_{01}}}\int_0^\infty \mathrm{d}y_2 y_2^{c_2-2}\,\mathrm{e}^{-\frac{y_2}{y_{02}}}\frac{y_1 y_2}{y_2-y_1}\left(\mathrm{e}^{-\gamma\frac{y_1 y_2}{y_2-y_1}}-1\right). \end{aligned} \tag{18}$$

3 The Bivariate Rayleigh-Gamma Distribution

The conditional bivariate Rayleigh distribution is given by:

$$P_{r_1 r_2}\left(r_1 r_2 / y_1 y_2\right) = \frac{4 r_1 r_2}{y_1 y_2 (1-\rho)} \mathrm{e}^{-\frac{1}{1-\rho}\left(\frac{r_1^2}{y_1}+\frac{r_2^2}{y_2}\right)} I_0\left(\frac{2\sqrt{\rho} r_1 r_2}{(1-\rho)\sqrt{y_1 y_2}}\right), \quad r_1, r_2 \geq 0. \quad (19)$$

Let us now define two random variables γ_1 and γ_2 in terms of r_1 and r_2 by the transformation:

$$\gamma_1 = r_1^2 \text{ and } \gamma_2 = r_2^2. \quad (20)$$

The Jacobian of this transformation is:

$$|J| = \frac{1}{4\sqrt{\gamma_1 \gamma_2}}.$$

The conditional squared bivariate Rayleigh distribution is:

$$P\gamma_1\gamma_2\left(\gamma_1\gamma_2 / y_1 y_2\right) = \frac{1}{y_1 y_2 (1-\rho)} \mathrm{e}^{-\frac{1}{1-\rho}\left(\frac{\gamma_1}{y_1}+\frac{\gamma_2}{y_2}\right)} I_0\left(\frac{2\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{(1-\rho)\sqrt{y_1 y_2}}\right). \quad (21)$$

Assuming that y_1 and y_2 are i.d. gamma RV's with parameters c and y_0, the joint pdf of y_1 and y_2 can be expressed as:

$$P y_1 y_2\left(y_1 y_2\right) = \frac{\rho_1^{\frac{c-1}{2}}}{\Gamma(c)(1-\rho) y_0^{c+1}} \left(y_1 y_2\right)^{\frac{c-1}{2}} \mathrm{e}^{-\frac{y_1+y_2}{y_0(1-\rho_1)}} I_{c-1}\left(\frac{\sqrt{4\rho_1 y_1 y_2}}{y_0(1-\rho_1)}\right), \quad (22)$$

where ρ_1 is the correlation between y_1 and y_2, and I_{c-1} is the modified Bessel function of the first kind of order $(c-1)$. By averaging (21) with respect to y_1 and y_2 it follows,

$$P\gamma_1\gamma_2\left(\gamma_1\gamma_2\right) = \int_0^\infty \mathrm{d} y_1 \int_0^\infty \mathrm{d} y_2 P\gamma_1\gamma_2\left(\gamma_1\gamma_2 / y_1 y_2\right) P y_1 y_2\left(y_1 y_2\right). \quad (23)$$

By substituting (21) and (22) in (23) it follows that:

$$\begin{aligned} P\gamma_1\gamma_2\left(\gamma_1\gamma_2\right) = &\int_0^\infty \mathrm{d} y_1 \int_0^\infty \mathrm{d} y_2 \frac{1}{y_1 y_2 (1-\rho)} \mathrm{e}^{-\frac{1}{(1-\rho)}\left(\frac{\gamma_1}{y_1}+\frac{\gamma_2}{y_2}\right)} I_0\left(\frac{2\sqrt{\rho \cdot \gamma_1\gamma_2}}{(1-\rho)\sqrt{y_1 y_2}}\right) \cdot \\ &\cdot \frac{\rho_1^{\frac{(c-1)}{2}}}{\Gamma(c)(1-\rho_1) y_0^{c+1}} \left(y_1 y_2\right)^{\frac{(c-1)}{2}} \mathrm{e}^{-\frac{y_1+y_2}{y_0(1-\rho_1)}} I_{c-1}\left(\frac{2\sqrt{\rho_1 y_1 y_2}}{y_0(1-\rho_1)}\right). \end{aligned} \quad (24)$$

The infinite series representation for $I_0(x)$ and $I_n(x)$ are:

$$I_0(x)=\sum_{i_1=0}^{\infty}\left(\frac{x}{2}\right)^{2i_1}\frac{1}{(i_1!)^2},\qquad I_n(x)=\sum_{i_2=0}^{\infty}\left(\frac{x}{2}\right)^{2i_2+n}\frac{1}{i_2!\Gamma(i_2+n+1)}. \tag{25}$$

Substituting (25) in (24) and changing the order of summations and integration can be written:

$$\begin{aligned}P\gamma_1\gamma_2\left(\gamma_1\gamma_2\right)&=\frac{1}{(1-\rho)}\cdot\frac{\rho_1^{\frac{(c-1)}{2}}}{\Gamma(c)(1-\rho_1)y_0^{c+1}}\cdot\\&\cdot\int_0^\infty \mathrm{d}y_1\int_0^\infty \mathrm{d}y_2\frac{1}{y_1y_2}\mathrm{e}^{-\frac{1}{(1-\rho)}\left(\frac{\gamma_1}{y_1}+\frac{\gamma_2}{y_2}\right)}\cdot\sum_{i_1=0}^{\infty}\frac{1}{(i_1!)^2}\left(\frac{\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{(1-\rho)}\right)^{2i_1}\cdot\\&\cdot\left(y_1y_2\right)^{-i_1+\frac{1}{2}(c-1)}\mathrm{e}^{-\frac{y_1+y_2}{y_0(1-\rho_1)}}\sum_{i_2=0}^{\infty}\frac{\left(y_1y_2\right)^{i_2+\frac{(c-1)}{2}}}{i_2!\Gamma(i_2+c)}\left(\frac{\sqrt{\rho_1}}{y_0\left(1-\rho_1\right)}\right)^{2i_2+c-1}=\\&=\frac{1}{\left(1-\rho\right)}\cdot\frac{\rho_1^{\frac{1}{2}(c-1)}}{\Gamma\left(c\right)\left(1-\rho_1\right)y_0^{c+1}}\sum_{i_1=0}^{\infty}\frac{1}{\left(i_1!\right)^2}\left(\frac{\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{\left(1-\rho\right)}\right)^{2i_1}\cdot\\&\cdot\sum_{i_2=0}^{\infty}\frac{1}{i_2!\Gamma\left(i_2+c\right)}\left(\frac{\sqrt{\rho_1}}{y_0\left(1-\rho_1\right)}\right)^{2i_2+c-1}\cdot\\&\cdot\int_0^\infty \mathrm{d}y_1\cdot y_1^{c-i_1+i_2-2}\,\mathrm{e}^{-\frac{\gamma_1}{(1-\rho)y_1}-\frac{y_1}{(1-\rho_1)y_0}}\int_0^\infty \mathrm{d}y_2\cdot y_2^{c-i_1+i_2-2}\,\mathrm{e}^{-\frac{\gamma_2}{(1-\rho)y_2}-\frac{y_2}{(1-\rho_1)y_0}}.\end{aligned} \tag{26}$$

$$\begin{aligned}P\gamma_1\gamma_2\left(\gamma_1\gamma_2\right)&=\frac{4\rho_1^{\frac{1}{2}(c-1)}}{\Gamma\left(c\right)\left(1-\rho\right)\left(1-\rho_1\right)y_0^{c+1}}\sum_{i_1=0}^{\infty}\frac{1}{\left(i_1!\right)^2}\left(\frac{\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{\left(1-\rho\right)}\right)^{2i_1}\cdot\\&\cdot\sum_{i_2=0}^{\infty}\frac{1}{i_2!\Gamma\left(i_2+c\right)}\left(\frac{\sqrt{\rho_1}}{y_0\left(1-\rho_1\right)}\right)^{2i_2+c-1}\cdot\left(\frac{\left(1-\rho_1\right)}{\left(1-\rho\right)}\frac{1}{y_0\sqrt{\gamma_1\gamma_2}}\right)^{c-i_1+i_2-1}\cdot\\&\cdot K_{c-i_1+i_2-1}\left(2\sqrt{\frac{\gamma_1}{\left(1-\rho\right)y_0\left(1-\rho_1\right)}}\right)K_{c-i_1+i_2-1}\left(2\sqrt{\frac{\gamma_2}{\left(1-\rho\right)y_0\left(1-\rho_1\right)}}\right).\end{aligned}$$

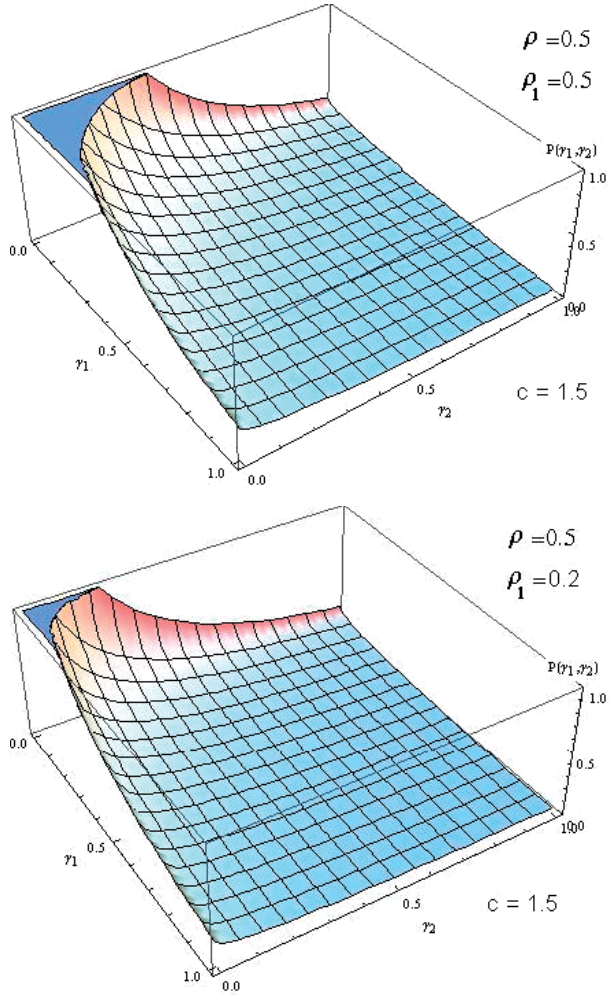

Fig. 1 – *The probability of bivariate Rayleigh-Gamma distribution.*

4 Nakagami – Gamma Distribution

The conditional Nakagami–m distribution has the term:

$$\Pr\left(r / y\right) = \frac{2}{\Gamma\left(m\right)}\left(\frac{m}{y}\right)^{m} r^{2m-1}\, \mathrm{e}^{-\frac{m}{y}r^{2}}, \quad r \geq 0, \tag{27}$$

where $\Gamma(m)$ is the Gamma function, y is an average signal power, and m an arbitrary fading security parameter with values from 0.5 through infinity. Transforming random variable r to γ, where is

$$\gamma = r^2,\ \ r = \sqrt{\gamma} \quad \text{and} \quad \frac{\mathrm{d}r}{\mathrm{d}\gamma} = \frac{1}{2\sqrt{\gamma}}.$$

we obtain probability density function for squared Nakagami–m random variable as follow:

$$P\gamma\left(\gamma / y\right) = \frac{2}{\Gamma\left(m\right)}\left(\frac{m}{y}\right)^m \gamma^{m-1}\,\mathrm{e}^{-\frac{m}{y}\gamma}, \quad \gamma \geq 0. \tag{28}$$

Averaging $P\gamma\left(\gamma / y\right)$ with respect to y, using (6), it follows:

$$\begin{aligned} P\gamma(\gamma) &= \int_0^\infty P\gamma\left(\gamma / y\right) Py(y)\,\mathrm{d}y = \\ &= \int_0^\infty \frac{2}{\Gamma(m)}\left(\frac{m}{y}\right)^m \gamma^{m-1}\,\mathrm{e}^{-\frac{m}{y}\gamma} \frac{{y_0}^{-c}}{\Gamma(c)} y^{c-1}\,\mathrm{e}^{-\frac{y}{y_0}}\,\mathrm{d}y = \\ &= \frac{{y_0}^{-c}}{\Gamma(c)\Gamma(m)} m^m \gamma^{m-1} \int_0^\infty \mathrm{d}y \cdot y^{-m+c-1}\,\mathrm{e}^{-\frac{m\gamma}{y} - \frac{y}{y_0}} = \\ &= \frac{{y_0}^{-c}}{\Gamma(c)\Gamma(m)} m^m \gamma^{m-1} \cdot 2\left(m y_0 \gamma\right)^{\frac{c-m}{2}} K_{c-m}\left(2\sqrt{\frac{m\gamma}{y_0}}\right). \end{aligned} \tag{29}$$

The cumulative distribution function of Nakagami-Gamma random variable is:

$$F\gamma(\gamma) = \int_0^\gamma P\gamma(x)\,\mathrm{d}x = \frac{{y_0}^{-c} m^m}{\Gamma(c)\Gamma(m)} \int_0^\gamma x^{m-1}\,\mathrm{d}x \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\,\mathrm{e}^{-\frac{mx}{y} - \frac{y}{y_0}}.$$

Changing the order of integrations, it follows:

$$\begin{aligned} F\gamma(\gamma) &= \frac{y_0^{-c} m^m}{\Gamma(c)\Gamma(m)} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\,\mathrm{e}^{-\frac{y}{y_0}} \int_0^\gamma x^{m-1}\,\mathrm{e}^{-\frac{mx}{y}}\,\mathrm{d}x = \\ &= \frac{y_0^{-c} m^m}{\Gamma(c)\Gamma\left(m\right)} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\,\mathrm{e}^{-\frac{y}{y_0}} \left(\frac{y}{m}\right)^m \left[\Gamma(m) - \gamma\left(m, \frac{mx}{y}\right)\right] = \\ &= 1 - \frac{{y_0}^{-c}}{\Gamma(c)\Gamma(m)} \int_0^\infty y^{c-1}\,\mathrm{e}^{-\frac{y}{y_0}}\,\gamma\left(m, \frac{mx}{y}\right)\mathrm{d}y. \end{aligned} \tag{30}$$

The MGF of Nakagami–m random variables is:

$$M\gamma(s) = \overline{\mathrm{e}^{\gamma s}} = \int_0^\infty \mathrm{e}^{\gamma s} P\gamma(\gamma)\,\mathrm{d}\gamma = \int_0^\infty \mathrm{d}\gamma \cdot \mathrm{e}^{\gamma s}\, \frac{y_0^{-c} m^m}{\Gamma(c)\Gamma(m)} \gamma^{m-1} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\, \mathrm{e}^{-\frac{m\gamma}{y} - \frac{y}{y_0}}.$$

Changing the order of integrations, we can write:

$$\begin{aligned} M\gamma(s) &= \frac{y_0^{-c}(m)^m}{\Gamma(c)\Gamma(m)} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1} e^{-\frac{y}{y_0}} \int_0^\infty \mathrm{e}^{\gamma s} \gamma^{m-1}\, \mathrm{e}^{-\frac{m\gamma}{y}}\, \mathrm{d}\gamma = \\ &= \frac{y_0^{-c}(m)^m}{\Gamma(c)\Gamma(m)} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\, \mathrm{e}^{-\frac{y}{y_0}} \left(\frac{y}{m - ys} \right)^m = \\ &= \frac{y_0^{-c}(m)^m}{\Gamma(c)\Gamma(m)} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\, \mathrm{e}^{-\frac{y}{y_0}} (m - ys)^{-m}. \end{aligned} \tag{31}$$

Moment of n-th order of Nakagami-m RV's is:

$$\overline{\gamma^n} = \int_0^\infty \gamma^n P\gamma(\gamma)\,\mathrm{d}\gamma = \int_0^\infty \gamma^n\, \mathrm{d}\gamma \frac{y_0^{-c} m^m}{\Gamma(c)\Gamma(m)} \gamma^{m-1} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\, \mathrm{e}^{-\frac{m\gamma}{y} - \frac{y}{y_0}}.$$

Changing the order of the integrations, it follows:

$$\begin{aligned} \overline{\gamma^n} &= \frac{y_0^{-c} m^m}{\Gamma(c)\Gamma(m)} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\, \mathrm{e}^{-\frac{y}{y_0}} \int_0^\infty \mathrm{d}\gamma \cdot \gamma^{n+m-1}\, \mathrm{e}^{-\frac{\gamma m}{y}} = \\ &= \frac{y_0^{-c} m^m}{\Gamma(c)\Gamma(m)} \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\, \mathrm{e}^{-\frac{y}{y_0}} \left(\frac{y}{m} \right)^{n+m} \Gamma(n+m) = \\ &= \frac{y_0^{-c} m^m}{\Gamma(c)\Gamma(m)} \Gamma(n+m) \int_0^\infty \mathrm{d}y \cdot y^{c-m-1}\, \mathrm{e}^{-\frac{y}{y_0}} = \\ &= \frac{\Gamma(n+m)\Gamma(c+n)}{\Gamma(c)\Gamma(m)} y_0^{\,n}. \end{aligned} \tag{32}$$

5 Bivariate Nakagami-Gamma Distribution

The bivariate Nakagami-Gamma distribution has the form:

$$\begin{aligned} &\Pr_1 r_2 \left(r_1 r_2 / y_1 y_2 \right) = \\ &= \frac{4m^{m+1}(r_1 r_2)^m}{\Gamma(m) y_1 y_2 (1-\rho)(y_1 y_2)^{\frac{1}{2}(m-1)}} \mathrm{e}^{-\frac{m}{1-\rho}\left(\frac{r_1^2}{y_1} + \frac{r_2^2}{y_2} \right)} I_{m-1}\left(\frac{2m\sqrt{\rho} r_1 r_2}{(1-\rho)\sqrt{y_1 y_2}} \right), \end{aligned} \tag{33}$$

where $y_1 = \overline{r_1^2}$, $y_2 = \overline{r_2^2}$, and $\rho = \dfrac{\operatorname{cov}\left(r_1^2 + r_2^2\right)}{\sqrt{\operatorname{var}\left(r_1^2\right) + \operatorname{var}\left(r_2^2\right)}}$ is the power correlation coefficient ($0 \le \rho < 1$).

Transformation of random variables r_1 and r_2 respectively into γ_1 and γ_2, gives $\gamma_1 = r_1^2$, $\gamma_2 = r_2^2$ and the Jacobian is:

$$|J| = \frac{1}{4\sqrt{\gamma_1\gamma_2}}.$$

We can obtain the joint probability density function of shared Nakagami-*m* RV's in the form:

$$P\gamma_1\gamma_2\left(\gamma_1\gamma_2 / y_1y_2\right) =$$

$$= \frac{m^{m+1}}{\Gamma(m)(1-\rho)(y_1y_2)^{\frac{m+1}{2}}}(\gamma_1\gamma_2)^{m-1}\,\mathrm{e}^{-\frac{m}{1-\rho}\left(\frac{\gamma_1}{y_1}+\frac{\gamma_2}{y_2}\right)} I_{m-1}\left(\frac{2m\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{(1-\rho)\sqrt{y_1y_2}}\right). \tag{34}$$

Averaging the expression (34) with respect to y_1 and y_2, it follows:

$$P\gamma_1\gamma_2\left(\gamma_1\gamma_2\right) = \int_0^\infty \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2 P\gamma_1\gamma_2\left(\gamma_1\gamma_2 / y_1y_2\right) Py_1y_2\left(y_1y_2\right) =$$

$$= \int_0^\infty \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2 \frac{m^{m+1}\left(\gamma_1\gamma_2\right)^{m-1}}{\Gamma(m)(1-\rho)(y_1y_2)^{\frac{m+1}{2}}}\,\mathrm{e}^{-\frac{m}{1-\rho}\left(\frac{\gamma_1}{y_1}+\frac{\gamma_2}{y_2}\right)} I_{m-1}\left(\frac{2m\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{(1-\rho)\sqrt{y_1y_2}}\right)\cdot \tag{35}$$

$$\cdot\frac{\rho_1^{\frac{c-1}{2}}\left(y_1y_2\right)^{\frac{1}{2}(c-1)}}{\Gamma(c)(1-\rho_1)y_0^{c+1}}\,\mathrm{e}^{-\frac{y_1+y_2}{y_0(1-\rho_1)}} I_{c-1}\left(\frac{2\sqrt{\rho_1}\sqrt{y_1y_2}}{y_0(1-\rho_1)}\right).$$

Using the infinite series representations for modified Bessel functions and changing the order of summation and integration, we can write:

$$P\gamma_1\gamma_2\left(\gamma_1\gamma_2\right) = \frac{m^{m+1}\left(\gamma_1\gamma_2\right)^{m-1}}{\Gamma(m)(1-\rho)} \int_0^\infty \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2 \left(y_1y_2\right)^{-\frac{1}{2}(m+1)}\,\mathrm{e}^{-\frac{m}{1-\rho}\left(\frac{\gamma_1}{y_1}+\frac{\gamma_2}{y_2}\right)}\cdot$$

$$\cdot\sum_{i_1=0}^{m-1}\frac{1}{i_1!\Gamma\left(i_1+m\right)}\left(\frac{m\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{(1-\rho)\sqrt{y_1y_2}}\right)^{2i_1+m-1}\cdot$$

$$\cdot\frac{\rho_1^{\frac{1}{2}(c-1)}\left(y_1y_2\right)^{\frac{1}{2}(c-1)}}{\Gamma\left(c\right)\left(1-\rho_1\right)y_0^{c+1}}\,\mathrm{e}^{-\frac{y_1+y_2}{y_0(1-\rho_1)}}\cdot\sum_{i_2=0}^{\infty}\frac{1}{i_2!\Gamma\left(i_2+c\right)}\left(\frac{\sqrt{\rho_1}\sqrt{y_1y_2}}{y_0\left(1-\rho_1\right)}\right)^{2i_2+c-1} =$$

$$=\frac{m^{m+1}\left(\gamma_1\gamma_2\right)^{m-1}}{\Gamma(m)(1-\rho)}\frac{\rho_1^{\frac{1}{2}(c-1)}}{\Gamma(c)\left(1-\rho_1\right)y_0^{c+1}}\sum_{i_1=0}^{\infty}\frac{1}{i_1!\Gamma\left(i_1+m\right)}\left(\frac{m\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{(1-\rho)}\right)^{2i_1+m-1}\cdot$$

$$\cdot\sum_{i_2=0}^{\infty}\frac{1}{i_2!\Gamma\left(i_2+c\right)}\left(\frac{\sqrt{\rho_1}}{y_0\left(1-\rho_1\right)}\right)^{2i_2+c-1}\cdot$$

$$\cdot\int_0^{\infty}\mathrm{d}\,y_1\cdot y_1^{-1-i_1+i_2+c-m}\,\mathrm{e}^{-\frac{m\gamma_1}{(1-\rho)y_1}-\frac{y_1}{(1-\rho_1)y_0}}\int_0^{\infty}\mathrm{d}\,y_2\cdot y_2^{-1-i_1+i_2+c-m}\,\mathrm{e}^{-\frac{m\gamma_2}{(1-\rho)y_2}-\frac{y_2}{(1-\rho_1)y_0}}$$

$$P\gamma_1\gamma_2\left(\gamma_1\gamma_2\right)=$$

$$=\frac{m^{m+1}\left(\gamma_1\gamma_2\right)^{m-1}}{\Gamma(m)\left(1-\rho\right)}\cdot\frac{4\rho_1^{\frac{1}{2}(c-1)}}{\Gamma(c)\left(1-\rho_1\right)y_0^{c+1}}\cdot\sum_{i_1=0}^{\infty}\frac{1}{i_1!\Gamma\left(i_1+m\right)}\left(\frac{m\sqrt{\rho}\sqrt{\gamma_1\gamma_2}}{(1-\rho)}\right)^{2i_1+m-1}\cdot$$

$$\cdot\sum_{i_2=0}^{\infty}\frac{1}{i_2!\Gamma\left(i_2+c\right)}\left(\frac{\sqrt{\rho_1}}{y_0\left(1-\rho_1\right)}\right)^{2i_2+c-1}\cdot\left(\frac{1-\rho}{1-\rho_1}\frac{1}{my_0\sqrt{\gamma_1\gamma_2}}\right)^{c-m-i_1+i_2}\cdot$$

$$\cdot K_{c-m-i_1+i_2}\left(2\sqrt{\frac{m\gamma_1}{(1-\rho)\left(1-\rho_1\right)y_0}}\right)K_{c-m-i_1+i_2}\left(2\sqrt{\frac{m\gamma_2}{(1-\rho)\left(1-\rho_1\right)y_0}}\right).$$

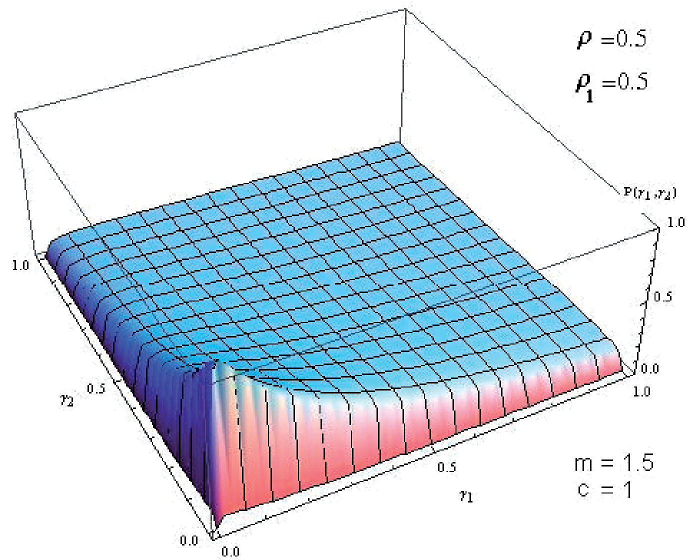

Fig. 2 – *The probability of bivariate Nakagami-Gamma distribution.*

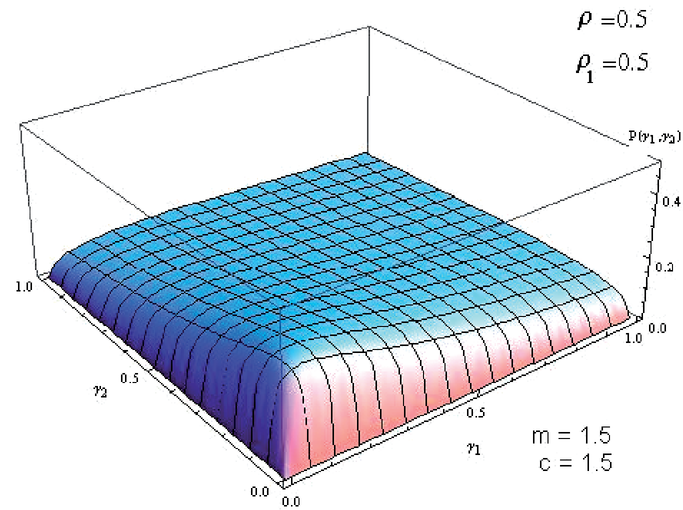

Fig. 3 – *The probability of bivariate Nakagami-Gamma distribution.*

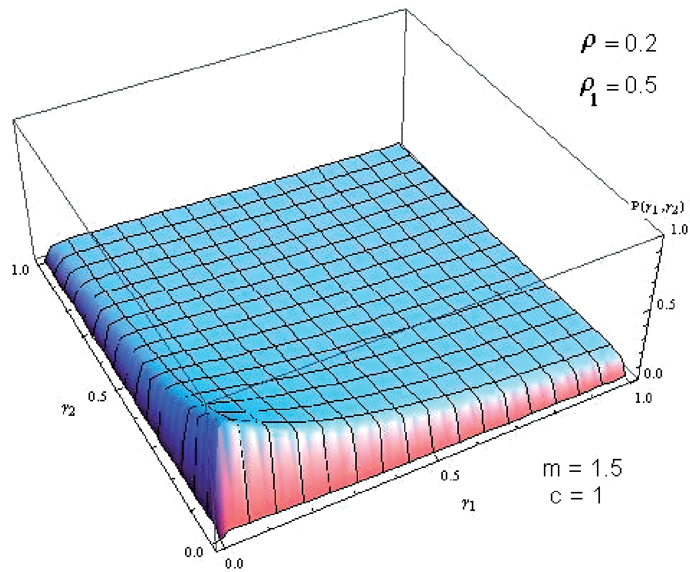

Fig. 4 – *The probability of bivariate Nakagami-Gamma distribution.*

Using the expression for the joint pdf of Nakagami RV's, we can determine the expressions for CDF, joint MGF and joint moments for Nakagami-*m* random variables.

6 Conclusion

This paper considers the Rayleigh-Gamma and Nakagami-Gamma distributions. Closed form of expressions for the probability density functions for Rayleigh-Gamma and Nakagami-Gamma random variables are obtained. In addition, the cumulative distribution functions, the generating functions and moments of these random variables are determined. These functions can be used to determine the outage probability and the bit error probability in digital communication systems operating over multipath and shadow fading channels. The Rayleigh distribution and shadow-fading channel are described with Rayleigh-Gamma distribution; the Nakagami-m distribution and shadow-fading channel are described with Nakagami-Gamma distribution. The results of this paper can be used to determine the performance analysis of macro- and micro-diversity systems. The macro-diversity systems are used to mitigate the shadowing and micro-diversity systems are used to mitigate the multipath fading.

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