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Statistical Properties for Multipath-Relayed Communications over Fading Channels

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Abstract: In this paper, a multipath transmission with no regenerative relays over fading channels has been addressed. The probability density function of minimum amplitude at relay stations input has been determined. The statistical results were subsequently applied to the analysis of the outage probability, the level crossing rate and the average outage duration.

Keywords: Average outage duration, Fading, Evel crossing rate, Outage probability, Probability density function, Telecommunication system, Threshold.

1 Introduction

The telecommunication system consists of the transmitter, several relay stations, and the receiver. The additive Gaussian noise, interference and fading influence both the input of receivers of all relay stations and at the one of the receiver station. The occurrence of fading at the relay and receiver station inputs can induce changes of the signal amplitude. The values will presumably fall below a predetermined value at any moment. The time intervals when we lost the connection may occur as well. Depending on the fading nature, we can determine the probability density function of the signal amplitude at relay stations and receiver inputs. The position of the relay stations should enable the occurrence the dominant component at the receiver input. If that be the case, the probability density function of signal amplitude has the Rician distribution. If the dominant component does not occur and is not visible line of sight between the pair of nearby relay stations does not exist, the probability density function of signal amplitude has the Rayleigh distribution. Except the Rayleigh and Rician distribution, the one of the signal amplitude and power in urban area can be well described by the Nakagami-m, Nakagami-q, Weibull or some other types of fading channels. Similarly, when the shadowing effect has been evidenced, fading can be described by the lognormal distribution. The fading occurring at the relay stations input causes changes in the signal amplitude.

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On relay telecommunication systems, the determination of the probability density function of signal amplitudes minimum is of particular scientific interest. This signal amplitudes minimum is very important for determining of the outage rate. Assuming that an outage is declared whenever signal amplitude falls below a predetermined threshold, the outage probability is determined by integrating the probability density function of minimum signal values below the threshold. The determination of joint probability density function of signal amplitudes minimum and their derivations are also the subject of this analysis. By employing this joint probability density function of minimum amplitudes the level crossing rate can be determined. The average outage duration is given as a quotient between the system outage probability and the joint level crossing rate. In order to obtain the joint probability density function of amplitudes minimum and their derivatives, we need to know the joint probability density function of signal amplitudes and their derivatives at the input of each receiving relay stations. The probability density function of amplitudes minimum in two or more time intervals has also been determined.

2 The Probability Density Function of Minimal Amplitude

There are several relay stations between transmitter and receiver, therefore the signal is transmitted over several sections. Fig. 1 represents the model of such system. Signals by the system sections are represented by $g_1, g_2, \dots g_n$.

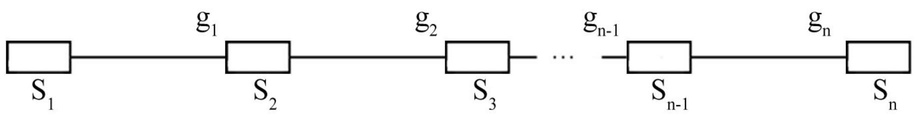

Fig.1 – *The model of the n-relay stations system.*

Probability density functions of signals $g_1, g_2, \dots g_n$ has the Rayleigh scenario

$$
\begin{aligned}
p_{g_1}(g_1) &= \frac{g_1}{\sigma_1^2} e^{-\frac{g_1^2}{2\sigma_1^2}}, \quad g_1 \ge 0 \\
p_{g_2}(g_2) &= \frac{g_2}{\sigma_2^2} e^{-\frac{g_2^2}{2\sigma_2^2}}, \quad g_2 \ge 0 \\
&\vdots \\
p_{g_n}(g_n) &= \frac{g_n}{\sigma_n^2} e^{-\frac{g_n^2}{2\sigma_n^2}}, \quad g_n \ge 0.
\end{aligned} \tag{1}
$$

The cumulative distributions of signals $g_1, g_2, \dots g_n$ are as it follows:

$$
\begin{aligned}
F_{g_1}(g_1) &= 1 - e^{-\frac{g_1^2}{2\sigma_1^2}}, g_1 \geq 0 \\
F_{g_2}(g_2) &= 1 - e^{-\frac{g_2^2}{2\sigma_2^2}}, g_2 \geq 0 \\
&\vdots \\
F_{g_n}(g_n) &= 1 - e^{-\frac{g_n^2}{2\sigma_n^2}}, g_n \geq 0.
\end{aligned} \tag{2}
$$

If the value of amplitude at one of sections is lower then the predetermined one threshold come the outage. The determination of this probability requires determining the probability density function of minimal amplitudes. Random variable g is defined as

$$g = \min\{g_1, g_2, \ldots, g_n\}. \tag{3}$$

Probability density of random variable g is

$$p_g(g) = \sum_{i=1}^{n} p_{g_i}(g) \prod_{\substack{j=1 \\ j\neq 1}}^{n} \left[1 - F_{g_j}(g)\right]. \tag{4}$$

The substitution results in

$$p_g(g) = \sum_{i=1}^{n} \frac{g_i}{\sigma_i^2} e^{-\frac{g_i^2}{2\sigma_i^2}} \prod_{\substack{j=1 \\ j\neq 1}}^{n} e^{-\frac{g_j^2}{2\sigma_j^2}}. \tag{5}$$

The definition of two RV's a and b as follows

$$
\begin{aligned}
a &= \min\{g_{11}, g_{12}, \ldots, g_{1n}\} \\
b &= \min\{g_{21}, g_{22}, \ldots, g_{2n}\}.
\end{aligned} \tag{6}
$$

The joint probability density function of RV's a and b can be written as

$$p_{ab}(a,b) = \sum_{i=1}^{n}\sum_{l=1}^{n} p_{g_{1i}g_{2l}}(a,b) \cdot \prod_{\substack{j=1 \\ j\neq 1}}^{n}\prod_{\substack{j=1 \\ j\neq 1}}^{n} \left[1 - F_{g_{1j}}(a)\right]\left[1 - F_{g_{2k}}(b)\right]. \tag{7}$$

Substitution gives

$$p_{ab}(a,b) = \sum_{i=1}^{n}\sum_{l=1}^{n} \frac{g_i g_l}{\sigma_i^2 \sigma_l^2} e^{-\frac{g_i^2}{2\sigma_i^2}-\frac{g_l^2}{2\sigma_l^2}} \cdot \prod_{\substack{j=1 \\ j\neq 1}}^{n}\prod_{\substack{j=1 \\ j\neq 1}}^{n} e^{-\frac{g_i^2}{2\sigma_i^2}-\frac{g_l^2}{2\sigma_l^2}}. \tag{8}$$

Fig. 2 represents a three-branch system model. The system comprises a transmitter and two relay stations.

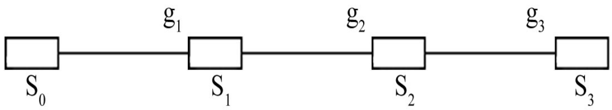

Fig. 2 – *A three-branch system model.*

Signals g_1, g_2 and g_3 are mutually dependent. These are determined by the joint probability density function

$$p_{g_1 g_2 g_3}\left(g_1, g_2, g_3\right), \tag{9}$$

where g is equal to minimal value of g_1, g_2 or g_3,

$$g = \min\left\{g_1, g_2, g_3\right\}. \tag{10}$$

As the signal g can have any value of g_1, g_2 or g_3, the probability density function of signal g is as follows:

$$\begin{aligned} p_g\left(g\right) = \int_g^{+\infty} \mathrm{d}g_2 \int_g^{+\infty} p_{g_1 g_2 g_3}\left(g_1, g_2, g_3\right) \mathrm{d}g_3 + \\ + \int_g^{+\infty} \mathrm{d}g_1 \int_g^{+\infty} p_{g_1 g_2 g_3}\left(g_1, g_2, g_3\right) \mathrm{d}g_3 + \int_g^{+\infty} \mathrm{d}g_1 \int_g^{+\infty} p_{g_1 g_2 g_3}\left(g_1, g_2, g_3\right) \mathrm{d}g_2. \end{aligned} \tag{11}$$

Fig. 3 represents a four-section system. Signals g_1, g_2, g_3 and g_4 are mutually dependent.

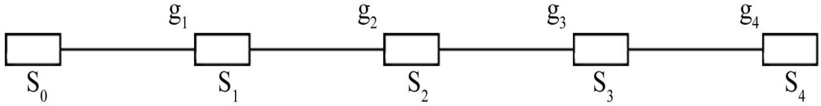

Fig. 3 – *A four- section system model.*

Joint probability density function of signals g_1, g_2, g_3 and g_4 are

$$p_{g_1 g_2 g_3 g_4}\left(g_1, g_2, g_3, g_4\right). \tag{12}$$

Signal g is

$$g = \min\left\{g_1, g_2, g_3, g_4\right\}. \tag{13}$$

If signal $g = g_1$ then $g_2 > g_1$, $g_3 > g_1$ and $g_4 > g_1$, and if signal $g = g_2$ then $g_1 > g_2$, $g_3 > g_2$ and $g_4 > g_2$, and if $g = g_3$ then $g_1 > g_3$, $g_3 > g_3$ and

$g_4 > g_3$, and if $g = g_4$ then $g_1 > g_4$, $g_2 > g_4$ and $g_3 > g_4$. In this case, probability density of random variable g is equal with

$$\begin{aligned} p_g(g) = & \int_g^{+\infty} \mathrm{d}g_2 \int_g^{+\infty} \mathrm{d}g_3 \int_g^{+\infty} p_{g_1 g_2 g_3 g_4}(g_1, g_2, g_3, g_4)\mathrm{d}g_4 + \\ & + \int_g^{+\infty} \mathrm{d}g_1 \int_g^{+\infty} \mathrm{d}g_{31} \int_g^{+\infty} p_{g_1 g_2 g_3 g_4}(g_1, g_2, g_3, g_4)\mathrm{d}g_4 + \\ & + \int_g^{+\infty} \mathrm{d}g_1 \int_g^{+\infty} \mathrm{d}g_2 \int_g^{+\infty} p_{g_1 g_2 g_3 g_4}(g_1, g_2, g_3, g_4)\mathrm{d}g_4 + \\ & + \int_g^{+\infty} \mathrm{d}g_1 \int_g^{+\infty} \mathrm{d}g_2 \int_g^{+\infty} p_{g_1 g_2 g_3 g_4}(g_1, g_2, g_3, g_4)\mathrm{d}g_3. \end{aligned} \tag{14}$$

The Fig. 4 represents a two-section system model.

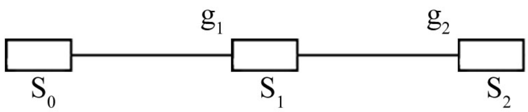

Fig. 4 – *Two-section system model.*

Signal g is

$$g = \min\{g_1, g_2\}. \tag{15}$$

Probability density of signal g is given by

$$p_g(g) = \int_g^{+\infty} p_{g_1 g_2}(g, g_2)\mathrm{d}g_2 + \int_g^{+\infty} p_{g_1 g_2}(g_2, g)\mathrm{d}g_1. \tag{16}$$

The Fig. 5 represents the n sections system model

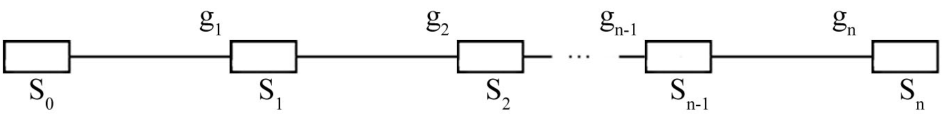

Fig. 5 – *An n sections system model.*

Fig. 6 represents a two- section system model.

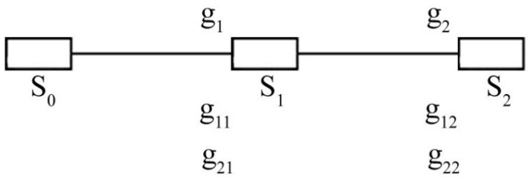

Fig. 6 – *A two- section system model.*

The signals at the section outputs are g_{11} and g_{12} in a particular time interval , whereas in the following time interval they are g_{21} and g_{22}. The signal a_1 has the minimal value of signals g_{11} and g_{12}, and a_2 is minimal value of g_{21} or g_{22},

$$\begin{aligned} a_1 &= \min\{g_{11}, g_{12}\} \\ a_2 &= \min\{g_{21}, g_{22}\}. \end{aligned} \tag{17}$$

Signals g_{11} and g_{12} are interdependent, whereas the signals g_{21} and g_{22} are dependent on and determinated by joint probability functions.

$$\begin{aligned} &p_{g_{11}g_{12}}(g_{11}, g_{12}) \\ &p_{g_{21}g_{22}}(g_{21}, g_{22}). \end{aligned} \tag{18}$$

Joint probability density function of signals a_1 and a_2 are

$$\begin{aligned} p_{a_1a_2}(a_1, a_2) = &\int_g^{+\infty} \mathrm{d}g_{12} \int_g^{+\infty} p_{g_{11}g_{12}}(a_1, g_{12})\, p_{g_{21}g_{22}}(a_2, g_{22})\,\mathrm{d}g_{22} + \\ &+\int_g^{+\infty} \mathrm{d}g_{12} \int_g^{+\infty} p_{g_{11}g_{12}}(a_1, g_{12})\, p_{g_{21}g_{22}}(g_{21}, a_2)\,\mathrm{d}g_{21} + \\ &+\int_g^{+\infty} \mathrm{d}g_{11} \int_g^{+\infty} p_{g_{11}g_{12}}(g_{11}, a_1)\, p_{g_{21}g_{22}}(a_2, g_{21})\,\mathrm{d}g_{22} + \\ &+\int_g^{+\infty} \mathrm{d}g_{11} \int_g^{+\infty} p_{g_{11}g_{12}}(g_{11}, a_1)\, p_{g_{21}g_{22}}(g_{21}, a_2)\,\mathrm{d}g_{21} \end{aligned} \tag{19}$$

Similarly, the signals g_{11}, g_{12}, g_{21} and g_{22} can be mutually dependent,

$$p_{g_{11}g_{12}g_{21}g_{22}}(g_{11}, g_{12}, g_{21}, g_{22}). \tag{20}$$

If that be the case, the joint probability density function of signals a_1 and a_2 equals

$$p_{a_1 a_2}(a_1,a_2) = \int_g^{+\infty} \mathrm{d}g_{12} \int_g^{+\infty} p_{g_{11}g_{12}g_{21}g_{22}}(a_1,g_{12},a_2,g_{22})\mathrm{d}g_{22} + \\ + \int_g^{+\infty} \mathrm{d}g_{12} \int_g^{+\infty} p_{g_{11}g_{12}g_{21}g_{22}}(a_1,g_{12},g_{21},a_2)\mathrm{d}g_{21} + \\ + \int_g^{+\infty} \mathrm{d}g_{11} \int_g^{+\infty} p_{g_{11}g_{12}g_{21}g_{22}}(g_{11},a_1,a_2,g_{22})\mathrm{d}g_{22} + \\ + \int_g^{+\infty} \mathrm{d}g_{11} \int_g^{+\infty} p_{g_{11}g_{12}g_{21}g_{22}}(g_{11},a_1,g_{21},a_2)\mathrm{d}g_{21} \tag{21}$$

3 Numerical Results

Probability density function of signal g is

$$p_g(g) = p_{g_1}(g)\int_g^{\infty} p_{g_2}(g_2)\mathrm{d}g_2 \int_g^{\infty} p_{g_3}(g_3)\mathrm{d}g_3 + \\ p_{g_2}(g)\int_g^{\infty} p_{g_1}(g_1)\mathrm{d}g_1 \int_g^{\infty} p_{g_3}(g_3)\mathrm{d}g_3 + \\ p_{g_3}(g)\int_g^{\infty} p_{g_1}(g_1)\mathrm{d}g_1 \int_g^{\infty} p_{g_2}(g_2)\mathrm{d}g_2. \tag{22}$$

If $p_{g_1}(g_1)$ has Nakagami-m, $p_{g_2}(g_2)$ has Rician and $p_{g_3}(g_3)$ has Rayleigh distribution,

$$p_{g_1}(g_1) = \frac{2}{\Gamma(m)}\left(\frac{m}{\rho}\right)^m g_1^{2m-1} e^{-\frac{m}{\rho}g_1^2} \\ p_{g_2}(g_2) = \frac{g_2}{\sigma^2} e^{-\frac{g_2^2+G^2}{2\sigma^2}} I_0\left(\frac{g_2 G}{\sigma_2^2}\right) \\ p_{g_3}(g_3) = \frac{g_3}{\sigma^2} e^{-\frac{g_3^2}{2\sigma^2}}. \tag{23}$$

Fig. 7 presents the graphics of probability density distribution for values $G = 2.5$, $m = 2$ and $\rho = 0.8$.

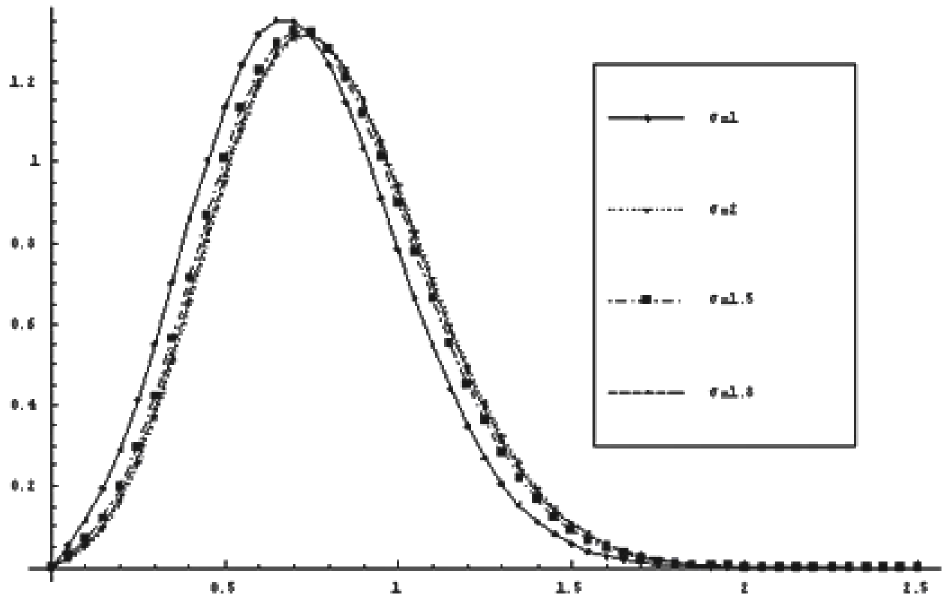

Fig. 7 – *The probability density function for* $G = 2.5$, $m = 2$ and $\rho = 0.8$.

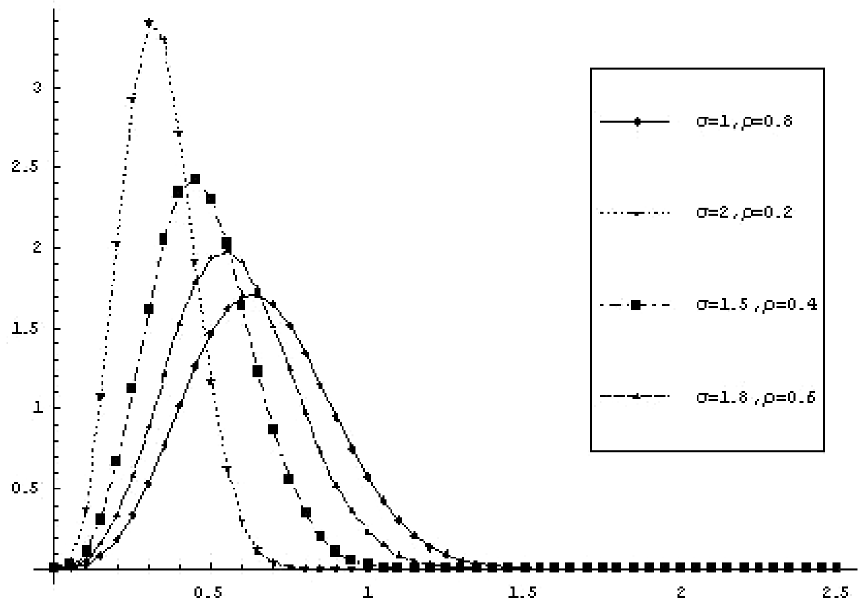

Fig. 8 – *The probability density function for* $G = 2.5$ *and* $m = 2$.

If $p_{g_1}(g_1)$ and $p_{g_2}(g_2)$ have Nakagami-m and $p_{g_3}(g_3)$ has Rician distribution

$$p_{g_1}(g_1)=\frac{2}{G(m)}\left(\frac{m}{r}\right)^m g_1^{2m-1}e^{-\frac{m}{r}g_1^2}$$

$$p_{g_2}(g_2)=\frac{2}{G(m)}\left(\frac{m}{r}\right)^m g_2^{2m-1}e^{-\frac{m}{r}g_2^2} \tag{24}$$

$$p_{g_3}(g_3)=\frac{g_3}{s^2}e^{-\frac{g_3^2+G^2}{2s_2^2}}I_0\left(\frac{g_3 G}{s^2}\right).$$

Fig. 8 represents the graphics of probability density function for $G=2.5$ and $m=2$.

4 The Probability Density Function and its Derivatives

Fig. 9 shows a two-section system model

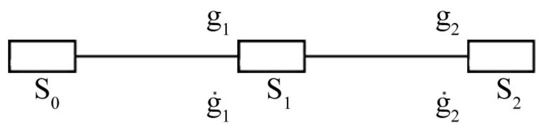

Fig. 9 – *A two-section system model.*

The joint probability density function of signal g_1 and its derivative $\dot{g}_1$ is

$$p_{g_1\dot{g}_1}(g_1,\dot{g}_1)=\frac{g_1}{\sigma^2}e^{-\frac{g_1^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{\dot{g}_1^2}{2\beta^2}}. \tag{25}$$

The joint probability density function of signal g_2 and its prime derivative $\dot{g}_2$ is

$$p_{g_2\dot{g}_2}(g_2,\dot{g}_2)=\frac{g_2}{\sigma^2}e^{-\frac{g_2^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{\dot{g}_2^2}{2\beta^2}}. \tag{26}$$

A signal a is defined as

$$a=\min\{g_1,g_2\}. \tag{27}$$

The joint probability density function of signal a and its prime derivative $\dot{a}$ is

$$p_{a\dot{a}}(a,\dot{a})=p_{g_1\dot{g}_1}(a,\dot{a})\left[1-F_{g_2}(a)\right]+p_{g_2\dot{g}_2}(a,\dot{a})\left[1-F_{g_1}(a)\right]. \tag{28}$$

The cumulative probability of signal g_1 is

$$F_{g_1}\left(g_1\right)=1-e^{-\frac{g_1^2}{2\sigma^2}}. \tag{29}$$

The cumulative probability of signal g_2 is

$$F_{g_2}\left(g_2\right)=1-e^{-\frac{g_2^2}{2\sigma^2}}. \tag{30}$$

The substitution gives the joint probability density function of signal a and its prime derivative $\dot{a}$,

$$p_{a\dot{a}}\left(a,\dot{a}\right)=2\frac{a}{\sigma^2}e^{-\frac{a^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi}\beta}e^{-\frac{\dot{a}}{2\beta^2}}e^{-\frac{a^2}{2\sigma^2}}. \tag{31}$$

5 The Probability Density Function in Two Time Intervals

Fig 10 represents a two-section system model.

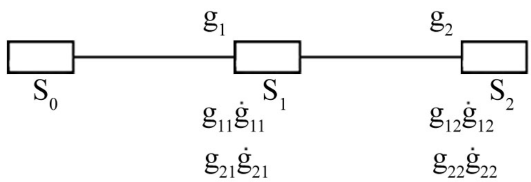

Fig. 10 – *Representation of a two-system model.*

The signals a_1 and a_2 are determined as

$$\begin{aligned} a_1 &= \min\left\{g_{11}, g_{12}\right\} \\ a_2 &= \min\left\{g_{21}, g_{22}\right\}. \end{aligned} \tag{32}$$

The joint probability density function of signals and their prime derivatives are

$$\begin{aligned} &p_{g_{11}\dot{g}_{11}}\left(g_{11},\dot{g}_{11}\right) \\ &p_{g_{12}\dot{g}_{12}}\left(g_{12},\dot{g}_{12}\right) \\ &p_{g_{21}\dot{g}_{21}}\left(g_{21},\dot{g}_{21}\right) \\ &p_{g_{22}\dot{g}_{22}}\left(g_{22},\dot{g}_{22}\right). \end{aligned} \tag{33}$$

The joint probability density function of signals a_1 and a_2, and their prime derivatives $\dot{a}_1$ and $\dot{a}_2$ are

$$\begin{aligned} p_{a_1a_2\dot{a}_1\dot{a}_2}\left(a_1,a_2,\dot{a}_1,\dot{a}_2\right) = {} & p_{g_{11}g_{21}\dot{g}_{11}\dot{g}_{21}}\left(a_1,a_2,\dot{a}_1,\dot{a}_2\right)\left[1-F_{g_{12}}\left(a_1\right)\right]\left[1-F_{g_{22}}\left(a_2\right)\right]+ \\ & +p_{g_{11}g_{22}\dot{g}_{11}\dot{g}_{22}}\left(a_1,a_2,\dot{a}_1,\dot{a}_2\right)\left[1-F_{g_{12}}\left(a_1\right)\right]\left[1-F_{g_{21}}\left(a_2\right)\right]+ \\ & +p_{g_{12}g_{21}\dot{g}_{12}\dot{g}_{21}}\left(a_1,a_2,\dot{a}_1,\dot{a}_2\right)\left[1-F_{g_{11}}\left(a_1\right)\right]\left[1-F_{g_{22}}\left(a_2\right)\right]+ \\ & +p_{g_{12}g_{22}\dot{g}_{12}\dot{g}_{22}}\left(a_1,a_2,\dot{a}_1,\dot{a}_2\right)\left[1-F_{g_{11}}\left(a_1\right)\right]\left[1-F_{g_{21}}\left(a_2\right)\right]. \end{aligned} \tag{34}$$

6 Conclusion

The signal transmission through relay stations has been addressed in the paper. A relay system comprises a transmitter, a receiver and several relay stations. Fading may occur at each relay station input. This phenomenon causes variability of signal amplitude at a receiver station input. Assuming that an outage of system be declared whenever an input signal falls below a predetermined threshold, the outage probability must be below predetermined value, which is a very important issue for designing digital telecommunication systems. The determination of the outage probability is required for determining a joint probability density function of signal amplitudes for all input sections of the transmission system. When the signal amplitude is at its minimum, the outage occurs when the minimum falls below the threshold. Integrating a probability density function of signal amplitude minimum from zero to a predetermined threshold value results in the outage probability.

Relay systems with two or several sections have been considered. The probability density function of signal at the receiving relay stations can be of Rayleigh, Racian, Nakagami-m, Nakagami-q or Weibull fading. When shadowing occurs, the probability density function is log-normal.

The designing of digital wireless systems requires the average outage duration as parameter. The average outage duration is determined as the quotient of the outage probability and average number of level crossing when amplitudes minimum falls below threshold. Therefore, only average number of the level crossing is needed to obtain the average outage duration. Determining the average number of level crossing is needed to determine joint probability density function of signal amplitudes minimum and theirs derivatives. That joint probability function is determined for several fading scenario combinations of interest (Rayleigh, Rician and Nakagami-m). By these combinations, signal amplitude minimum and their derivatives are independent. Derivatives of these amplitudes have Gaussian probability density function. In addition, joint probability density function of minimum amplitudes in two different time intervals is determined. That joint probability density function determines the outage probability in two time intervals. There are some numerical results in closed-form expressions for probability density function of minimum signal amplitudes at inputs of receiving stations and their graphic representations. These results can be useful for designing digital telecommunication systems. Determining a number of relay stations and distance between sections for predetermined values of the outage probability is possible.

7 Reference

[1] M.O. Hasna, M.S. Alouini: End to End Performance of Transmission Systems with Relays over Rayleigh Fading Channels, IEEE Trans. Wireless Commun., Vol.2, No. 6, Nov 2003, pp. 1126 – 1161.

[2] M.O. Hasna, M.S. Alouini: Harmonic Mean and End to End Performance of Transmission Systems with Relays, IEEE Trans. Commun., Vol. 52, No. 1, Jan. 2004, pp. 130 – 135.

[3] M.O. Hasna, M.S. Alouini: Outage Probability of Multihop Transmission over Nakagami Fading Channels, IEEE Commun. Lett., Vol. 7, No. 5, May 2003, pp. 216 – 218.

[4] M.K. Simon, M.S. Alouini: Digital Communication over Fading Channels 2nd ed., John Wiley & Sons, New York, 2005.