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Simple Exponential Stability Criteria of Linear Discrete Time-Delay Systems

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Abstract: In this paper, new delay-independent asymptotic and exponential stability conditions of linear discrete delay systems based on the Lyapunov-Krasovskii method have been derived. A numerical example has been developed so as to show applicability of the derived results.

Keywords: Discrete systems, Time delay, Exponential stability, Delay independent criteria.

1 Introduction

The problem of stability analysis and controller design for time-delay systems have been given considerable attention over the past decades. The existing stabilization results for time delay systems can be classified into two types, i.e. delay independent stabilization [1-4, 23, 25] and delay-dependent stabilization [5-9, 22, 25]. The delay-independent stabilization provides a controller which stabilizes a system irrespective of the extent of the delay. On the other hand, the delay-dependent stabilization is concerned with the size of the delay which usually provides an upper bound of the delay capable of ensuring the stability for any delay lower than the upper bound.

As most physical systems occur in continuous time, consequently the theories for stability analysis and controller synthesis are mainly developed for the continuous time. However, it is more feasible that a discrete-time approach is used for the purpose, as the controller is usually digitally implemented. Despite this significance, discrete-time systems with delays: [10-18] have not been paid due attention. It is mainly due to the fact that the delay-difference equations with known delays can be converted into a higher-order smaller-delay system by augmentation approach. However, in systems with great delay extent, this scheme will lead to large-dimensional systems. Furthermore, for systems with unknown delay the augmentation scheme is not applicable.

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In this paper, new delay-independent asymptotic and exponential stability conditions are presented for discrete state-delayed systems. These conditions are derived by Lyapunov - Krasovskii method for discrete time-delay systems.

2 Notation and Preliminaries

Review of used notations has been presented in **Table 1**.

Table 1

Used notations

$\mathbb{R}$	Real vector space
$\mathbb{Z}^+$	Positive integer
F	Real matrix
I	Identity matrix
F^T	Transpose of matrix F
$F>0$	Positive definite matrix
$F\geq 0$	Positive semi definite matrix
$\lambda(F)$	Eigenvalue of matrix F
$\sigma(F)=\lVert F\rVert$	Singular value of matrix
$\lVert F\rVert=\sqrt{\lambda_{\max}\left(A^TA\right)}$	Euclidean matrix norm of F

A linear, autonomous, multivariable discrete time-delay system can be represented by the difference equation

$$\boldsymbol{x}(k+1)=A_0\boldsymbol{x}(k)+A_1\boldsymbol{x}(k-h) \tag{1}$$

with an associated function of initial state

$$\boldsymbol{x}(\theta)=\psi(\theta),\, \theta\in\{-h,-h+1,\dots,0\}\triangleq\Delta \tag{2}$$

where

$$x(k)=x(k,\Psi)\in\mathbb{R}^n,\ \forall\ \theta\in\Delta,\ \forall\ k\in\mathbb{Z}^+,$$

$$x_k\triangleq\{x(k-h),x(k-h+1),\dots,x(k)\}=x(k+\theta)$$

is state vector, A_0 and $A_1\in\mathbb{R}^{n\times n}$ are constant matrices of appropriate dimension and $h\in\mathbb{Z}^+$ is unknown time delay in general case. If $\mathbb{D}\left(\Delta,\mathbb{R}^n\right)$ is space of functions mapping the discrete interval Δ into $\mathbb{R}^n$, then, $x_k\in\mathbb{D}$, $\mathbb{D}\ni\phi(\theta):\ \Delta\mapsto\mathbb{R}^n$, $\lVert\phi\rVert_D\triangleq\sup_{\theta\in\Delta}\lVert\phi(\theta)\rVert$ is the norm of an element $\phi\in\mathbb{D}$ in $\mathbb{D}$ and $f:\mathbb{D}\to\mathbb{R}^n$. If $\mathbb{D}^\gamma=\left\{\phi\in\mathbb{D}:\lVert\phi\rVert_D<\gamma,\gamma\in\mathbb{R}\right\}\subset\mathbb{D}$.

Definition 1. The equilibrium state $x=0$ of system (1) is globally asymptotically stable if any initial $\psi(\theta)$ which satisfies

$$\psi(\theta)\in\mathbb{D}^{\infty} \tag{3}$$

holds

$$\lim_{k\to\infty}\boldsymbol{x}(k,\psi)\to 0. \tag{4}$$

Theorem 1. [25] If there are positive numbers α and β and continuous functional $V:\mathbb{D}\to\mathbb{R}$ such that

$$0<V(x_k)\le\alpha\|x_k\|_D^2,\quad \forall x_k\ne 0,\quad V(0)=0 \tag{5}$$

$$\Delta V(x_k)\triangleq V(x_{k+1})-V(x_k)\le-\beta\|x(k)\|^2 \tag{6}$$

$\forall x_k\in\mathbb{D}$ satisfying (1) then the solution $x=0$ of equation (1) and (2) is global asymptotically stable.

Definition 2. Discrete system with time delay (1) is globally asymptotically stable if and only if its the solution $x=0$ is global asymptotically stable.

Lemma 1. For any two matrices F and G of $n\times m$ dimension and for any square matrix $P=P^T>0$ of dimension *n*, the following statement is true

$$(F+G)^T P(F+G)\le(1+\varepsilon)F^T PF+\left(1+\varepsilon^{-1}\right)G^T PG, \tag{7}$$

where ε is a positive constant.

3 Main Results

Theorem 2. If for any given matrix $Q=Q^T>0$ there is matrix $P=P^T>0$, being such that the following matrix equation is fulfilled

$$\left(1+\frac{\|A_1\|_2}{\|A_0\|_2}\right)A_0^{\,T}PA_0+\left(1+\frac{\|A_0\|_2}{\|A_1\|_2}\right)A_1^{\,T}PA_1-P=-Q. \tag{8}$$

Then the system (1) with $\|A_0\|_2\ne 0$ and $\|A_1\|_2\ne 0$ is asymptotically stable.

Proof: As the Lyapunov functional is

$$V(\boldsymbol{x}_k)=\boldsymbol{x}^T(k)P\boldsymbol{x}(k)+\sum_{j=1}^{h}\boldsymbol{x}^T(k-j)S\boldsymbol{x}(k-j)\quad P=P^T>0,\ S=S^T\ge 0, \tag{9}$$

where

$$\boldsymbol{x}_k = \boldsymbol{x}(k+\theta),\ \theta \in \{-h, -h+1, \dots, 0\}. \tag{10}$$

The forward difference along the solutions of system (1) is

$$\begin{aligned}\Delta V(\boldsymbol{x}_k) = &\left[A_0\boldsymbol{x}(k) + A_1\boldsymbol{x}(k-h)\right]^T P\left[A_0\boldsymbol{x}(k) + A_1\boldsymbol{x}(k-h)\right] \\ &- \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)S\boldsymbol{x}(k) - \boldsymbol{x}^T(k-h)S\boldsymbol{x}(k-h).\end{aligned} \tag{11}$$

Applying **Lemma 1** on (11), one can get

$$\begin{aligned}\Delta V(\boldsymbol{x}_k) \le &(1+\varepsilon)\boldsymbol{x}^T(k)A_0^T P A_0 \boldsymbol{x}(k) \\ &+ \left(1+\varepsilon^{-1}\right)\boldsymbol{x}^T(k-h)A_1^T P A_1 \boldsymbol{x}(k-h) \\ &- \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)S\boldsymbol{x}(k) \\ &- \boldsymbol{x}^T(k-h)S\boldsymbol{x}(k-h)\end{aligned} \tag{12}$$

or

$$\begin{aligned}\Delta V(\boldsymbol{x}_k) \le &\boldsymbol{x}^T(k)\left[(1+\varepsilon)A_0^T P A_0 + S - P\right]\boldsymbol{x}(k) \\ &+ \boldsymbol{x}^T(k-h)\left[\left(1+\varepsilon^{-1}\right)A_1^T P A_1 - S\right]\boldsymbol{x}(k-h).\end{aligned} \tag{13}$$

If one adopts

$$S = \left(1+\varepsilon^{-1}\right)A_1^T P A_1, \tag{14}$$

then

$$\Delta V(\boldsymbol{x}_k) \le \boldsymbol{x}^T(k)\left[(1+\varepsilon)A_0^T P A_0 + \left(1+\varepsilon^{-1}\right)A_1^T P A_1 - P\right]\boldsymbol{x}(k). \tag{15}$$

Let us define the following function

$$f(\varepsilon, \boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\left[(1+\varepsilon)A_0^T P A_0 + \left(1+\varepsilon^{-1}\right)A_1^T P A_1\right]\boldsymbol{x}(k) \tag{16}$$

Since both matrices $A_0^T P A_0$ and $A_1^T P A_1$ are symmetric and positive semi-definite, then, based on *Raleigh* and *Amir-Moez* inequalities, [19, 20], the following can be stated:

$$\begin{aligned}f(\varepsilon, \boldsymbol{x}(k)) &\le \boldsymbol{x}^T(k)\left[(1+\varepsilon)\lambda_{\max}\left(A_0^T P A_0\right) + \left(1+\varepsilon^{-1}\right)\lambda_{\max}\left(A_1^T P A_1\right)\right]\boldsymbol{x}(k) \\ &\le \boldsymbol{x}^T(k)\left[(1+\varepsilon)\lambda_{\max}(P)\lambda_{\max}\left(A_0^T A_0\right) + \left(1+\varepsilon^{-1}\right)\lambda_{\max}(P)\lambda_{\max}\left(A_1^T A_1\right)\right]\boldsymbol{x}(k) \\ &= \lambda_{\max}(P)\boldsymbol{x}^T(k)\left[(1+\varepsilon)\sigma^2_{\max}(A_0) + \left(1+\varepsilon^{-1}\right)\sigma^2_{\max}(A_1)\right]\boldsymbol{x}(k) \\ &= g(\varepsilon)\lambda_{\max}(P)\left\|\boldsymbol{x}(k)\right\|_2^2.\end{aligned} \tag{17}$$

Scalar function

$$g(\varepsilon)=(1+\varepsilon)\sigma^2_{\max}(A_0)+(1+\varepsilon^{-1})\sigma^2_{\max}(A_1) \tag{18}$$

possesses its minimum at

$$\varepsilon_{\min}=\frac{\sigma_{\max}(A_1)}{\sigma_{\max}(A_0)}=\frac{\|A_1\|_2}{\|A_0\|_2}, \tag{19}$$

from where

$$f(\varepsilon_{\min},\boldsymbol{x}(k))\le f(\varepsilon,\boldsymbol{x}(k)). \tag{20}$$

As

$$\begin{aligned}\Delta V(\boldsymbol{x}_k)&\le \boldsymbol{x}^T(k)\left[\lambda_{\max}(P)g(\varepsilon_{\min})I_n-P\right]\boldsymbol{x}(k)\\&\le \boldsymbol{x}^T(k)\left[\lambda_{\max}(P)g(\varepsilon)I_n-P\right]\boldsymbol{x}(k)\end{aligned} \tag{21}$$

$\varepsilon_{\min}$ can be replaced by ε in (15), whereby

$$\begin{aligned}\Delta V(\boldsymbol{x}_k)&\le \boldsymbol{x}^T(k)\left[(1+\varepsilon_{\min})A_0^TPA_0+(1+\varepsilon_{\min}^{-1})A_1^TPA_1-P\right]\boldsymbol{x}(k)\\&\le \boldsymbol{x}^T(k)\left[(1+\varepsilon_{\min})A_0^TPA_0+(1+\varepsilon_{\min}^{-1})A_1^TPA_1-P\right]\boldsymbol{x}(k).\end{aligned} \tag{22}$$

If the condition (8) is satisfied then

$$\Delta V(x_k)\le\lambda_{\min}\{Q\}\|x(k)\|^2=-\beta\|x(k)\|^2<0,\quad \beta\hat{=}\lambda_{\min}\{Q\}>0. \tag{23}$$

Likewise, for $x_k\neq 0$ holds

$$\begin{aligned}0<V(x_k)&\le\max\left\{x^T(k)Px(k)+(1+\varepsilon_{\min}^{-1})\sum_{j=1}^{h}x^T(k-j)A_1^TPA_1x(k-j)\right\}\le\\&\le\left[\lambda_{\max}\{P\}+h(1+\varepsilon_{\min}^{-1})\lambda_{\max}\{A_1^TPA_1\}\right]\|x(k)\|_D^2=\alpha\|x(k)\|_D^2,\end{aligned} \tag{24}$$

$$\alpha\hat{=}\lambda_{\max}\{P\}+h(1+\varepsilon_{\min}^{-1})\lambda_{\max}\{A_1^TPA_1\}>0,$$

thus based on **Theorem 1**, system (1) is asymptotically stable.

Definition 3. [24] The system (1) is said to have a stability degree α (or to be exponentially stable), with $\alpha>1$, being real positive scalar, if the state of system given (1) can be written as:

$$x(k)=\alpha^{-k}p(k) \tag{25}$$

and the system governing the state $p(k)$ is globally asymptotically stable. In this case, the parameter α is called the convergence rate (see [10] for continuous case).

Theorem 3. If for any given matrix $Q = Q^T > 0$ there is the matrix $P = P^T > 0$ such that the following matrix equation is fulfilled

$$\alpha^2 \left(1 + \frac{\|A_1\|_2}{\|A_0\|_2}\right) A_0{}^T P A_0 + \alpha^{2(h+1)} \left(1 + \frac{\|A_0\|_2}{\|A_1\|_2}\right) A_1{}^T P A_1 - P = -Q , \qquad (26)$$

then system (1) with $\|A_0\|_2 \neq 0$ and $\|A_1\|_2 \neq 0$ is exponentially stable, the convergence rate being α.

Proof: Let us define a new variable [24]:

$$p(k) = x(k)\alpha^k . \qquad (27)$$

Then from (1) follows:

$$p(k+1) = \alpha A_0 p(k) + \alpha^{h+1} A_1 p(k-h) . \qquad (28)$$

Also, the following can be defined:

$$\hat{A}_0 = \alpha A_0, \quad \hat{A}_1 = \alpha^{h+1} A_1 \qquad (29)$$

then

$$p(k+1) = \hat{A}_0 p(k) + \hat{A}_1 p(k-h) . \qquad (30)$$

The Lyapunov equation (26) is obtained by applying the result of **Theorem 2** on (30).

4 Numerical Example

Example 1

Let us consider a discrete delay system described by

$$\boldsymbol{x}(k+1) = A_0 \boldsymbol{x}(k) + A_1 \boldsymbol{x}(k-h)$$

$$A_0 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & a \end{bmatrix}, \; A_1 = \wp \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix}$$

where $\wp$ stands for the adjustable parameter and system scalar parameter, a takes the values of –0.15 and 0.5.

The delay-independent asymptotic stability conditions are characterized by means of range of the parameter $\wp$. These are summarized in **Table 2**.

Theorem 2 provides results highly close to the stability boundary. Therefore, the derived results are quite precise.

Table 2

Stability conditions in respect of the parameter $\wp$.

Parameter a	-0.15	$+0.50$
Theorem 2	$\lvert\wp\rvert < 2.09$	$\lvert\wp\rvert < 1.51$
Stability boundary	$\lvert\wp\rvert = 2.11$	$\lvert\wp\rvert = 1.52$

Applying **Theorem 3** for $a = 0.5$, $\wp = 1.2$, $h = 2$ and $\alpha = 1.1$ leads to the conclusion that this system is exponentially stable.

5 Conclusion

In this paper, based on Lyapunov-Krasovskii method, new sufficient conditions for delay-independent asymptotic and exponential stability of linear discrete delay systems are presented. Numerical examples are presented to demonstrate the applicability of the present approach.

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