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A Lyapunov-Krasovskii Methodology for Asymptotic Stability of Discrete Time Delay Systems

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Abstract: This paper presents a Lyapunov-Krasovskii methodology for asymptotic stability of discrete time delay systems. Based on the methods, delay-independent stability condition is derived. A numerical example has been working out to show the applicability of results derived.

Keywords: Time delay systems, Lyapunov-Krasovskii method, Asymptotic stability.

1 Introduction

During the last decades, considerable attention has been devoted to the problem of stability analysis and controller design for time-delay systems. The existing stabilization results for time delay systems can be classified into two types: delay independent stabilization [1-4] and delay-dependent stabilization [5-9]. The delay-independent stabilization provides a controller which can stabilize a system irrespective of the size of the delay. On the other hand, the delay dependent stabilization is concerned with the size of the delay and usually provides an upper bound of the delay such that the closed-loop system is stable for any delay less than the upper bound.

Since most physical systems evolve in continuous time, it is natural that theories for stability analysis and controller synthesis are mainly developed for continuous-time. However, it is more reasonable that one should use a discrete-time approach for that purpose because the controller is usually implemented digitally. Despite this significance mentioned, less attention has been paid to discrete-time systems with delays: [10-18]. It is mainly due to the fact that the delay-difference equations with known delays can be converted into a higher-order delay less system by augmentation approach. However, for systems with

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large known delay amounts, this scheme will lead to large-dimensional systems. Furthermore, for systems with unknown delay the augmentation scheme is not applicable.

The use of Lyapunov methods for the stability analysis of time-delay systems has been an ever growing subject of interest starting with the pioneering works of Krasovskii [19] and Repin [20]. Recently, in [11-18, 21] modified Lyapunov–Krasovskiı functionals were introduced for which the time derivative includes terms which not only depend on the present but also on the past states of the delay system. This modification allows using the functionals for robustness analysis of time delay systems.

However, to the best of our knowledge, very small number of papers are investigated Lyapunov-Krasovskii method for discrete systems. Elaydi and Yhang [22] are developing a general theory of stability for nonlinear finite delay difference equations: a Lyapunov-Krasovskii and Razumikhin method. In [23] Lyapunov-Krasovskii method for discrete time delay systems with descriptor model transformation has been considered, while [24] is examined Lyapunov-Krasovskii method for neutral nonlinear discrete time delay systems.

In this paper, we give some extensions of Lyapunov-Krasovskii method for retarded functional differential equation (continuous time delay systems) [25] to discrete time delay systems. Our aim had been to develop a new simple general theory of stability of autonomous discrete time delay systems expressing as counterpart to Lyapunov-Krasovskii method for continuous time delay systems proposed in [25]. The obtained theorems hold for both constant and time varying delays. The theorems are simple mathematical forms according to [23-24] and can apply to autonomous discrete time delay systems. From these theorems can be carry out various stability conditions for both linear and nonlinear time delay systems.

The rest of this paper is organized as follows. In Section 2, we introduce our notation and preliminaries. Then in Section 3 we develop Lyapunov-Krasovskii type asymptotic stability theorems for discrete delay systems. In Section IV, as the theoretic application, the Lyapunov-Krasovskii type asymptotic stability result is applied to some kinds of discrete delay systems and a simple delay-independent stability condition is derived in form linear matrix inequality. Also, simple example is given to illustrate the obtained results.

2 Notation and Preliminaries

$\mathbb{R}$	Real vector space
$\mathbb{Z}^{+}$	Positive integer
F	Real matrix

I Identity matrix
F^T Transpose of matrix F
$F>0$ Positive definite matrix
$F \geq 0$ Positive semi definite matrix
$\lambda(F)$ Eigenvalue of matrix F
$\sigma(F) = \| F \|$ Singular value of matrix
$\|F\| = \sqrt{\lambda_{\max}\left(A^T A\right)}$ Euclidean matrix norm of F

A autonomous, multivariable discrete time-delay system can be represented by the difference equation

$$x(k+1) = f(x_k) \tag{1}$$

with an associated function of initial state

$$x(\theta) = \psi(\theta), \theta \in \{-h, -h+1, \dots, 0\} \triangleq \Delta . \tag{2}$$

Where

$$x(k) = x(k,\psi) \in \mathbb{R}^n$$
$$\forall\theta \in \Delta, \forall k \in \mathbb{Z}^+, x_k \triangleq \{x(k-h), x(k-h+1), \ldots, x(k)\} = x(k+\theta)$$

is state vector, $A \in \mathbb{R}^{n \times n}$ is a constant matrix of appropriate dimension and $h \in \mathbb{Z}^+$ is unknown time delay in general case. Let $\mathbb{D}\left(\Delta, \mathbb{R}^n\right)$ is space of functions mapping the discrete interval Δ into $\mathbb{R}^n$. Then, $x_k \in \mathbb{D}$, $\mathbb{D} \ni \phi(\theta)$: $\Delta \mapsto \mathbb{R}^n$, $\|\phi\|_D \triangleq \sup_{\theta \in \Delta} \|\phi(\theta)\|$ is the norm of an element $\phi \in \mathbb{D}$ in $\mathbb{D}$ and $f : \mathbb{D} \to \mathbb{R}^n$. Let $\mathbb{D}^\gamma = \left\{\phi \in \mathbb{D} : \|\phi\|_D < \gamma, \gamma \in \mathbb{R}\right\} \subset \mathbb{D}$.

3 Main Results

In sequel, we give the general Lyapunov-Krasovskii methods for discrete time delay systems as counterpart to Lyapunov-Krasovskii methods for continuous time delay systems proposed in [25].

Definition 1. The equilibrium state $x = 0$ of system (1) is asymptotically stable if any initial $\psi(\theta)$ which satisfies

$$\psi(\theta) \in \mathbb{D}^\infty \tag{3}$$

holds

$$\lim_{k\to\infty} x(k,\psi) \to 0 . \tag{4}$$

Theorem 1. If there exist continuous functional $V:\mathbb{D}\to\mathbb{R}$ and continuous nondecreasing functions v and w: $\mathbb{R}^+\to\mathbb{R}^+$ with features $v(0)=w(0)=0$, $v(s)>0$ and $w(s)>0$ $\forall s>0$, such that

$$0<V(x_k)\le v\left(\left\|x_k\right\|_D\right), \quad V(0)=0 , \tag{5}$$

$$\Delta V(x_k) \triangleq V(x_{k+1})-V(x_k)\le -w\left(\left\|x_k\right\|_D\right), \tag{6}$$

$\forall x_k \in\mathbb{D}$ satisfying (1), then the solution $x=0$ of equations (1) and (2) is asymptotically stable.

Proof. From (6) follows

$$\sum_{j=0}^{k}\Delta V(x_j)=V(x_{k+1})-V(x_0)\le -\sum_{j=0}^{k} w\left(\left\|x_j\right\|_D\right) \tag{7}$$

and from $V\left(x_k\right)>0$ and (7) hold

$$V(x_0)\ge V(x_{k+1})+\sum_{j=0}^{k} w\left(\left\|x_j\right\|_D\right)\ge \sum_{j=0}^{k} w\left(\left\|x_j\right\|_D\right)\ge w\left(\left\|x_k\right\|_D\right). \tag{8}$$

Using second inequality in (5), and inequalities (8) hold

$$w\left(\left\|x_k\right\|_D\right)\le \sum_{j=0}^{k} w\left(\left\|x_j\right\|_D\right)\le V(x_0)\le v\left(\left\|x_0\right\|_D\right)=v\left(\left\|\psi(\theta)\right\|_D\right). \tag{9}$$

Based on features of functions v and w and $\forall\psi(\theta)\in\mathbb{D}^\infty$ following

$$\begin{gathered}
\left\|\psi(\theta)\right\|_D<\infty,\\
v\left(\left\|\psi(\theta)\right\|_D\right)<\infty,\\
\lim_{k\mapsto\infty}\sum_{j=0}^{k} w\left(\left\|x_j\right\|_D\right)<\infty,\\
\lim_{k\mapsto\infty}\left\|x_k\right\|_D=0,\\
\lim_{k\mapsto\infty}\left\|x(k)\right\|=0,\\
\lim_{k\mapsto\infty}x(k)=0,
\end{gathered} \tag{10}$$

i.e. system (1) is asymptotically stable.

Theorem 2. If there exist positive numbers α and β and continuous functional $V:\mathbb{D}\to\mathbb{R}$ such that

$$0 < V(x_k) \le \alpha \|x_k\|_D^2, \quad \forall x_k \ne 0, \quad V(0) = 0, \tag{11}$$

$$\Delta V(x_k) \triangleq V(x_{k+1}) - V(x_k) \le -\beta \|x_k\|_D^2, \tag{12}$$

$\forall x_k \in \mathbb{D}$ satisfying (1) then the solution $x = 0$ of equation (1) - (2) is asymptotically stable.

Proof. The proof follows from proof of *Theorem* 1 adopting $v(s) = \alpha s^2$ and $w(s) = \beta s^2$.

A difficulty in applying *Theorem* 1 and 2 consists in the facts that in practice, one often obtains upper bounds on $\Delta V(x_k)$ which only depend on $\|x(k)\|$. For such cases, the following theorems are useful.

Theorem 3. If there exist continuous functional $V:\mathbb{D}\to\mathbb{R}$ and continuous nondecreasing functions v and w: $\mathbb{R}^+ \to \mathbb{R}^+$ with features $v(0) = w(0) = 0$, $v(s) > 0$ and $w(s) > 0$ $\forall s > 0$, such that

$$0 < V(x_k) \le v\left(\|x_k\|_D\right), \quad V(0) = 0, \tag{13}$$

$$\Delta V(x_k) \triangleq V(x_{k+1}) - V(x_k) \le -w\left(\|x(k)\|\right), \tag{14}$$

$\forall x_k \in \mathbb{D}$ satisfying (1), then the solution $x = 0$ of equations (1) and (2) is asymptotically stable.

Proof. The proof follows from proof of *Theorem* 1 considering inequality $\|x_k\|_D \ge \|x(k)\|$ i.e. $w\left(\|x_k\|_D\right) \ge w\left(\|x(k)\|\right)$.

Theorem 4. If there exist positive numbers α and β and continuous functional $V:\mathbb{D}\to\mathbb{R}$ such that

$$0 < V(x_k) \le \alpha \|x_k\|_D^2, \quad \forall x_k \ne 0, \quad V(0) = 0, \tag{15}$$

$$\Delta V(x_k) \triangleq V(x_{k+1}) - V(x_k) \le -\beta \|x(k)\|^2, \tag{16}$$

$\forall x_k \in \mathbb{D}$ satisfying (1) then the solution $x = 0$ of equations (1) and (2) is asymptotically stable.

Proof. The proof follows from proof of *Theorem* 3 adopting $v(s)=\alpha s^2$ and $w(s)=\beta s^2$.

Definition 2. Discrete system with time delay (1) is asymptotically stable if and only if it's the solution $x=0$ is asymptotically stable.

4 Aplication and Numerical Example

Previous results can use for derive simple stability criteria for discrete system with time delay

$$x(k+1)=A_0x(k)+A_1x(k-h). \tag{17}$$

For example, the following lemma presents one such result.

Lemma 1. The discrete time-delay system (1) is asymptotically stable if there exist matrices $P>0$ and $Q>0$ such that following linear matrix inequality (LMI) hold

$$\begin{bmatrix} Q-P & 0 & A_0{}^TP \\ (*) & -Q & A_1{}^TP \\ (*) & (*) & -P \end{bmatrix}<0. \tag{18}$$

Proof. Let the Lyapunov functional be

$$V(x_k)=x^T(k)Px(k)+\sum_{j=1}^{h}x^T(k-j)Qx(k-j),\quad P=P^T>0,\quad Q=Q^T>0. \tag{19}$$

The forward difference along the solutions of system (1) is

$$\begin{aligned}\Delta V(k)&=\left[A_0x(k)+A_1x(k-h)\right]^T P\left[A_0x(k)+A_1x(k-h)\right]-\\ &-x^T(k)Px(k)+x^T(k)Qx(k)-x^T(k-h)Qx(k-h)=\\ &=\begin{bmatrix} x(k) \\ x(k-h)\end{bmatrix}^T\begin{bmatrix} A_0{}^TPA_0-P+Q & A_0{}^TPA_1 \\ (*) & A_1{}^TPA_1-Q\end{bmatrix}\begin{bmatrix} x(k) \\ x(k-h)\end{bmatrix}.\end{aligned} \tag{20}$$

If the following equation is satisfied

$$\Sigma \triangleq \begin{bmatrix} A_0{}^TPA_0-P+Q & A_0{}^TPA_1 \\ (*) & A_1{}^TPA_1-Q\end{bmatrix}<0, \tag{21}$$

then

$$\begin{bmatrix} A_0^T P A_0 - P + Q & A_0^T P A_1 \\ (*) & A_1^T P A_1 - Q \end{bmatrix} = \begin{bmatrix} Q - P & 0 \\ (*) & -Q \end{bmatrix} + \begin{bmatrix} A_0^T P A_0 & A_0^T P A_1 \\ (*) & A_1^T P A_1 \end{bmatrix} \\ = \begin{bmatrix} Q - P & 0 \\ (*) & -Q \end{bmatrix} + \begin{bmatrix} A_0^T \\ A_1^T \end{bmatrix} P \begin{bmatrix} A_0 & A_1 \end{bmatrix} < 0 \quad (22)$$

Using Schur complements, [26], it is easy to see that the condition (21) is equivalent to

$$\begin{bmatrix} Q - P & 0 & A_0^T \\ (*) & -Q & A_1^T \\ (*) & (*) & -P^{-1} \end{bmatrix} < 0 . \quad (23)$$

Note that the condition (23) is not LMI condition due to the existence of the term $-P^{-1}$. Pre and post multiply (23) with $dig\{I, I, P\}$ we obtain LMI condition (18).

If the condition (18) is satisfied then

$$\Delta V(x_k) \le -\lambda_{\min}\{\Sigma\} \left\| \begin{bmatrix} x(k) \\ x(k-h) \end{bmatrix} \right\|^2 = -\lambda_{\min}\{\Sigma\} \left[\|x(k)\|^2 + \|x(k-h)\|^2 \right] \\ \le -\lambda_{\min}\{\Sigma\} \|x(k)\|^2 = -\beta \|x(k)\|^2 , \qquad \beta = \lambda_{\min}\{\Sigma\} \quad (24)$$

Likewise, for $x_k \ne 0$ holds

$$0 < V(x_k) \le \max\left\{ x^T(k) P x(k) + \sum_{j=1}^{h} x^T(k-j) Q x(k-j) \right\} \\ \le \left[\lambda_{\max}\{P\} + h\lambda_{\max}\{Q\} \right] \|x(k)\|_D^2 = \alpha \|x(k)\|_D^2 \\ \alpha \triangleq \lambda_{\max}\{P\} + h\lambda_{\max}\{A_1^T P A_1\} > 0 \quad (25)$$

so, based on *Theorem* 4, system (1) is asymptotically stable.

I sequel, we give simple example to illustrate the previous result (18).

Example 1. Let us consider a linear discrete delay system described by

$$x(k+1) = A_0 x(k) + A_1 x(k-h),$$

$$A_0 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & a \end{bmatrix}, \; A_1 = \wp \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix},$$

where $\wp$ is adjustable parameter and system scalar parameter a takes the following values: -0.15 and 0.5.

The **delay-independent** asymptotic stability conditions are characterized by means of range of parameter $\wp$ and are summarized in **Table 1**. For $Q = I_2$, *Lemma* 1 give results corresponding the stability boundary. This appears that performed results have insignificant conservation.

Table 1

Stability conditions.

Parameter a	- 0.15	+ 0.50
Lemma 1	$\|\wp\| < 2.11$	$\|\wp\| < 1.51$
Stability boundary	$\|\wp\| = 2.11$	$\|\wp\| = 1.51$

5 Conclusion

In this paper, we give the general Lyapunov-Krasovskii methods for discrete time delay systems as counterpart to Lyapunov-Krasovskii methods for continuous time delay systems proposed in [25]. Based on these methods, a simple delay-independent stability condition is derived. Numerical examples are presented to demonstrate the applicability of the present approach.

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