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Mixed Method of Model Reduction for Uncertain Systems

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Abstract: A mixed method for reducing a higher order uncertain system to a stable reduced order one is proposed. Interval arithmetic is used to construct a generalized Routh table for determining the denominator polynomial of the reduced system. The reduced numerator polynomial is obtained using factor division method and the steady state error is minimized using gain correction factor. The proposed method is illustrated using a numerical example.

Keywords: Model Reduction, Uncertain systems, Factor division, Gain correction factor

1 Introduction

The analysis and design of practical control systems become complex when the order of the system increases. Therefore, to analyze such systems, it is necessary to reduce it to a lower order system, which is a sufficient representation of the higher order system.

In recent decades, much effort has been made in the field of model reduction for fixed systems and several methods like: Aggregation method [1], Pade approximation [12], Routh approximation [7], Moment matching technique [13], and L^{∞} optimization technique [6] have been proposed. Among them Routh approximation technique has been recognized as the most powerful method because of its ability to yield stable reduced models for stable high-order systems [2].

In general, the practical systems have uncertainties about its parameters. Thus practical systems will have coefficients that may vary and it is represented by interval. Interval arithmetic such as addition, subtraction, multiplication and division are discussed in [3, 5]. In literature [2, 3] the authors presented model reduction techniques for higher order uncertain system. The limitations of the above method are discussed in [8]. A generalized method for constructing the

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Routh table of interval polynomial is proposed in [14] which overcomes some of the limitations of [2, 3].

In this paper, the method proposed in [4, 14] is integrated with factor division method [11] for obtaining the stable reduced order model. Also a gain correction factor is used to improve the steady state characteristic for interval systems.

2 Problem Formulation

Consider a higher order linear SISO uncertain system represented by the transfer function as

$$G(s)=\frac{N(s)}{D(s)},$$

$$G(s)=\frac{\left[c_0^-,c_0^+\right]+\left[c_1^-,c_1^+\right]s+\left[c_2^-,c_2^+\right]s^2+\cdots+\left[c_{n-1}^-,c_{n-1}^+\right]s^{n-1}}{\left[d_0^-,d_0^+\right]+\left[d_1^-,d_1^+\right]s+\left[d_2^-,d_2^+\right]s^2+\cdots+\left[d_n^-,d_n^+\right]s^n}, \quad (1)$$

where $\left[c_i^-,c_i^+\right]$, $i=0,1,2,\dots,n-1$ and $\left[d_i^-,d_i^+\right]$, $i=0,1,2,\dots,n$ are the interval coefficients of higher order numerator and denominator polynomials respectively.

The corresponding k^{th} order reduced model is

$$R(s)=\frac{N_k(s)}{D_k(s)},$$

$$R(s)=\frac{\left[r_0^-,r_0^+\right]+\left[r_1^-,r_1^+\right]s+\left[r_2^-,r_2^+\right]s^2+\cdots+\left[r_{k-1}^-,r_{k-1}^+\right]s^{k-1}}{\left[b_0^-,b_0^+\right]+\left[b_1^-,b_1^+\right]s+\left[b_2^-,b_2^+\right]s^2+\cdots+\left[b_k^-,b_k^+\right]s^k}, \quad (2)$$

where $\left[r_i^-,r_i^+\right]$, $i=0,1,2,\dots,k-1$ and $\left[b_i^-,b_i^+\right]$, $i=0,1,2,\dots,k$ are the interval coefficients of lower order numerator and denominator polynomials, respectively.

The problem is to reduce the system of form (1) to the system of the form (2) such that the lower order system mimics the higher order one as closely as possible.

2.2 Determination of reduced denominator

Routh table is constructed for the denominator of the higher order system using the algorithm proposed by Dolgin [4, 14]. The reduced denominator is obtained by direct truncation of the elements in Routh table. Consider the denominator polynomial of a higher order system

$$D(s)=\left[d_0^-,d_0^+\right]+\left[d_1^-,d_1^+\right]s+\left[d_2^-,d_2^+\right]s^2+\cdots+\left[d_n^-,d_n^+\right]s^n .$$

Corresponding generalized Routh table is

$$\begin{array}{llll}
s^n & \left[d_n^-,d_n^+\right] & \left[d_{n-2}^-,d_{n-2}^+\right] & \left[d_{n-4}^-,d_{n-4}^+\right] \; \cdots \\
 & =\left[d_{10}^-,d_{10}^+\right] & \left[d_{11}^-,d_{11}^+\right] & \left[d_{12}^-,d_{12}^+\right] \; \cdots \\
s^{n-1} & \left[d_{n-1}^-,d_{n-1}^+\right] & \left[d_{n-3}^-,d_{n-3}^+\right] & \left[d_{n-5}^-,d_{n-5}^+\right] \; \cdots \\
 & =\left[d_{20}^-,d_{20}^+\right] & \left[d_{21}^-,d_{21}^+\right] & \left[d_{22}^-,d_{22}^+\right] \; \cdots \\
s^{n-2} & \left[d_{30}^-,d_{30}^+\right] & \left[d_{31}^-,d_{31}^+\right] & \left[d_{32}^-,d_{32}^+\right] \; \cdots \\
\vdots & & & \\
s^1 & \left[d_{n-1,0}^-,d_{n-1,0}^+\right] & & \\
s^0 & \left[d_{n,0}^-,d_{n,0}^+\right] & &
\end{array}$$

From the Table, the lower order polynomials can be obtained. This algorithm offers stable reduced order models for a given stable higher order system.

2.2 Determination of reduced numerator

The nth order original system given in equation (1) is equated to the k^{th} order reduced model with unknown parameters represented by equation (2). Hence,

$$\begin{aligned}
&\frac{\left[c_0^-,c_0^+\right]+\left[c_1^-,c_1^+\right]s+\cdots+\left[c_{n-1}^-,c_{n-1}^+\right]s^{n-1}}{\left[d_0^-,d_0^+\right]+\left[d_1^-,d_1^+\right]s+\cdots+\left[d_n^-,d_n^+\right]s^n}=\\
&=\frac{\left[r_0^-,r_0^+\right]+\left[r_1^-,r_1^+\right]s+\cdots+\left[r_{k-1}^-,r_{k-1}^+\right]s^{k-1}}{\left[b_0^-,b_0^+\right]+\left[b_1^-,b_1^+\right]s+\cdots+\left[b_k^-,b_k^+\right]s^k}.
\end{aligned} \tag{3}$$

Rewriting the equation (3), we obtain

$$\begin{aligned}
&\left(\left[c_0^-,c_0^+\right]\cdot\left[b_0^-,b_0^+\right]\right)+\left(\left[c_1^-,c_1^+\right]\cdot\left[b_0^-,b_0^+\right]+\left[c_0^-,c_0^+\right]\cdot\left[b_1^-,b_1^+\right]\right)s+\cdots\\
&\cdots+\left(\left[c_{n-1}^-,c_{n-1}^+\right]\cdot\left[b_k^-,b_k^+\right]\right)s^{k-1+n}=\left(\left[r_0^-,r_0^+\right]\cdot\left[d_0^-,d_0^+\right]\right)+\\
&+\left(\left[r_1^-,r_1^+\right]\cdot\left[d_0^-,d_0^+\right]+\left[r_0^-,r_0^+\right]\cdot\left[d_1^-,d_1^+\right]\right)s+\cdots\\
&\cdots+\left(\left[r_{k-1}^-,r_{k-1}^+\right]\cdot\left[d_n^-,d_n^+\right]\right)s^{k-1+n}.
\end{aligned} \tag{4}$$

Equating the coefficients of the corresponding terms in the equation (4), the following relations are obtained:

$$
\begin{aligned}
&\left(\left[r_0^-, r_0^+\right]\cdot\left[d_0^-, d_0^+\right]\right) = \left(\left[c_0^-, c_0^+\right]\cdot\left[b_0^-, b_0^+\right]\right),\\
&\left(\left[r_0^-, r_0^+\right]\cdot\left[d_1^-, d_1^+\right] + \left[r_1^-, r_1^+\right]\cdot\left[d_0^-, d_0^+\right]\right) = \left(\left[c_0^-, c_0^+\right]\cdot\left[b_1^-, b_1^+\right] + \left[c_1^-, c_1^+\right]\cdot\left[b_0^-, b_0^+\right]\right), \qquad (5)\\
&\vdots\\
&\left(\left[r_{k-1}^-, r_{k-1}^+\right]\cdot\left[d_n^-, d_n^+\right]\right) = \left(\left[c_{n-1}^-, c_{n-1}^+\right]\cdot\left[b_k^-, b_k^+\right]\right).
\end{aligned}
$$

Comparing the first k relations of the equation (5) i.e., comparing the coefficients of $s^0, s^1, \ldots, s^{k-1}$, unknown coefficients $\left[r_i^-, r_i^+\right]$, $i = 0,1,2,...,k-1$ are determined.

To minimize the steady state error the 'Zeros' are adjusted by multiplying the numerator polynomial with the gain correction factor η. It can be calculated using the relation

$$\eta = \left.\frac{G(s)}{R(s)}\right|_{s=0}.$$

For interval systems, η is calculated after converting the interval coefficients of $G(s)$ and $R(s)$ into the fixed coefficients by taking their means. Thus the gain correction factor is

$$\eta = \left(\frac{c_0}{d_0}\right)\left(\frac{b_0}{r_0}\right), \qquad (6)$$

where

$$c_0 = \frac{c_0^- + c_0^+}{2},\ b_0 = \frac{b_0^- + b_0^+}{2},\ d_0 = \frac{d_0^- + d_0^+}{2} \text{ and } r_0 = \frac{r_0^- + r_0^+}{2}.$$

3 Simulation Results

Consider the seventh order system given by the transfer function

$$G(s) = \frac{N(s)}{D(s)},$$

$$
\begin{aligned}
D(s) = {} & [0.95, 1.05]s^7 + [8.779, 9.703]s^6 + [52.231, 57.729]s^5 + \\
& + [182.875, 202.125]s^4 + [429.02, 474.18]s^3 + \\
& + [572.47, 632.73]s^2 + [325.28, 359.52]s + [57.352, 63.389].
\end{aligned}
$$

$$
\begin{aligned}
N(s) = {} & [1.9, 2.1]s^6 + [24.7, 27.3]s^5 + [157.7, 174.3]s^4 + \\
& + [541.975, 599.025]s^3 + [929.955, 1027.845]s^2 + \\
& + [721.81, 797.79]s + [187.055, 206.745].
\end{aligned}
$$

To obtain denominator polynomial, the generalized Routh table is constructed for $D(s)$

s^7	$[0.95, 1.05]$	$[52.231, 57.729]$	$[429.02, 474.18]$	$[325.28, 359.52]$
s^6	$[8.779, 9.703]$	$[190, 194.98]$	$[582.23, 622.97]$	$[57.352, 63.389]$
s^5	$[31.67, 36.63]$	$[384.43, 388.35]$	$[319.834, 351.9]$	
s^4	$[86.2, 90.126]$	$[510.5, 513.32]$	$[57.352, 63.389]$	
s^3	$[186.87, 189.7]$	$[311.5, 313.46]$		
s^2	$[364.72, 366.62]$	$[59.74, 61]$		
s^1	$[281.08, 282.35]$			
s^0	$[59.74, 61]$			

By direct truncation, from above table the second order denominator polynomial is

$$D_2(s) = [364.72, 366.62]s^2 + [281.08, 282.35]s + [59.74, 61].$$

To obtain second order reduced numerator polynomial,

$$G(s) = R(s),$$

$$\frac{[187.055, 206.745] + [721.81, 797.79]s + \cdots + [1.9, 2.1]s^6}{[57.352, 63.389] + [325.28, 359.52]s + \cdots + [0.95, 1.05]s^7} =$$

$$= \frac{[r_0^-, r_0^+] + [r_1^-, r_1^+]s}{[364.72, 366.62]s^2 + [281.08, 282.35]s + [59.74, 61]}.$$

Rewriting,

$$\left([187.055, 206.745]\cdot[0.083, 0.161]\right)+$$
$$+\left([721.81, 797.79]\cdot[59.74, 61]+[187.055, 206.745]\times[281.08, 282.35]\right)s+$$
$$+\cdots+\left([1.9, 2.1]\cdot[364.72, 366.62]\right)s^8=\left(\left[r_0^-, r_0^+\right]\cdot[57.352, 63.389]\right)+$$
$$+\left(\left[r_1^-, r_1^+\right]\cdot[57.352, 63.389]+\left[r_0^-, r_0^+\right]\cdot[325.28, 359.52]\right)s+\cdots$$
$$\cdots+\left(\left[r_1^-, r_1^+\right]\cdot[0.95, 1.05]\right)s^8.$$

Comparing the coefficients of and, s^0 and s^1

$$\left[r_0^-, r_0^+\right]=[176.29, 219.9]$$
$$\left[r_1^-, r_1^+\right]=[262.53, 866.53].$$

The reduced numerator polynomial is

$$N_2(s)=\eta.\left([262.53, 866.53]s+[176.29, 219.9]\right).$$

The gain correction factor found using (6) is $\eta = 0.994$. Thus, the reduced numerator polynomial is

$$N_2(s)=[260.955, 861.331]s+[175.232, 218.581].$$

The second order reduced system for the higher order system $G(s)$ is obtained as

$$R(s)=\frac{[260.955, 861.331]s+[175.232, 218.581]}{[364.72, 366.62]s^2+[281.08, 282.35]s+[59.74, 61]}.$$

The γ table for $D(s)$ formed by the algorithm proposed in [2] is obtained as

$[57.35, 63.69]$	$[572.47, 632.75]$	$[182.88, 202.13]$	$[8.78, 9.703]$
$[325.28, 359.52]$	$[429.02, 474.18]$	$[52.23, 57.73]$	$[0.95, 1.05]$
$[434.35, 623.69]$	$[155.28, 214.2]$	$[7.759, 10.56]$	
$[175.3, 564.55]$	$[30.29, 77.08]$	$[0.662, 1.51]$	
$[-36.94, 614.78]$	$[0.741, 32.37]$		

The second order system obtained by method [2] is

$$R_2(s)=\frac{[1.16, 1.84]s+[0.27, 0.53]}{s^2+[0.52, 0.83]s+[0.08, 0.16]}.$$

It is noted that the lower bound of the interval entry $\left[d_{50}^{-}, d_{50}^{+}\right]$ of the γ table is negative, thus restricting the completion of the table. Hence reduced-order interval polynomials of degree four or greater cannot be obtained by [2].

The proposed method guarantees stability for a stable higher order system and thus any lower order model can be derived with good accuracy. Also it may be noted that the proposed method involves less mathematical complexity compared to algorithm [2], where both the γ and δ tables need to be formulated, which increases the complexity and requires a great deal of computational effort.

Fig. 1 and Fig. 2, show step response and frequency response of higher and reduced order transfer function. From the responses, it can be observed that the reduced order system is stable and it closely matches with the original system at all frequencies (in Fig. 2). Moreover, it is found that the proposed method and the method proposed in [2] match almost closely.

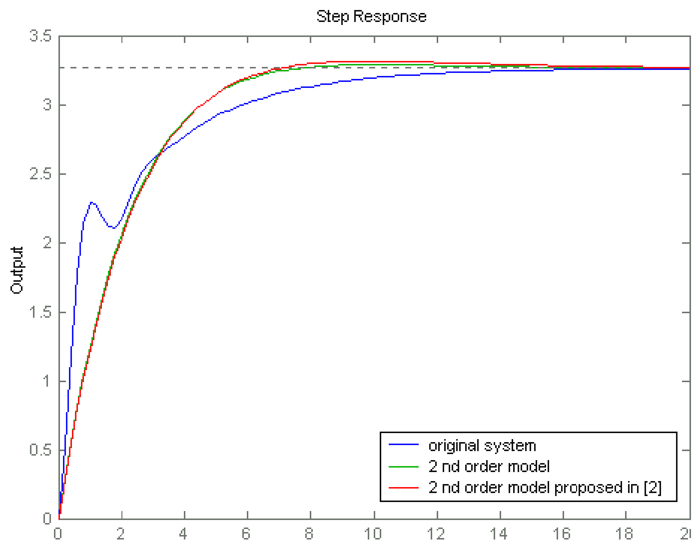

Fig. 1 - *Step response of the original and the reduced order models.*

But, system dynamic error always exists when a higher order system is reduced. The quantitative criteria for measuring the performance are chosen as Integral Square Error (ISE), Integral Absolute Error (IAE) and Integral of Time weighted Absolute Error (ITAE).

$$\text{ISE}=\int e^2\,\mathrm{d}t, \qquad \text{IAE}=\int |e|\,\mathrm{d}t, \qquad \text{ITAE}=\int t\,|e|\,\mathrm{d}t.$$

ISE, IAE accounts mainly for errors at the beginning of the response and to a lower degree for the steady state deviation. ITAE takes account of the error at the beginning but also emphasizes the steady state. For the reduced models (proposed method and [2]) and the original system, the Performance Indices (ISE, IAE and ITAE) are tabulated in **Table 1** and are found to be matching closely.

Table 1

Performance Indices for higher order and reduced systems for 10s.

	$G(s)$	$R(s)$	$R_2(s)$ by [2]
ISE	32.19	34.47	34.73
IAE	98.87	103.8	104.3
ITAE	17.3	17.29	17.32

Kharitonov polynomials were constructed for the reduced model and their robust stability was verified (in Appendix I). Also the dominant poles for the higher order system G(s), proposed reduced order system R(s) and R(s) by [2] are $-0.314, -0.385 \pm 0.129i$ and $-0.338 \pm 0.0954i$ respectively which are closely matching. Thus the transient response of the higher order system is not much affected by this reduction method.

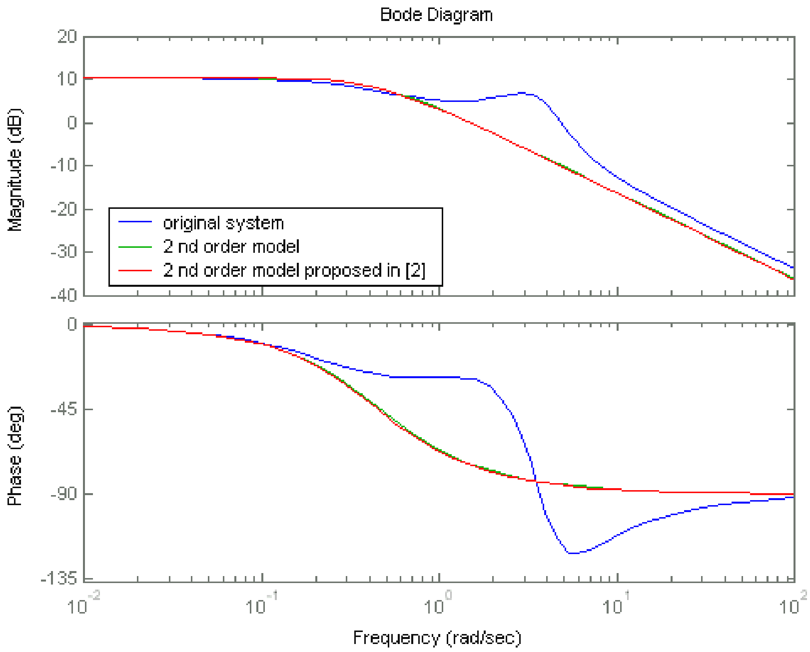

Fig. 2 - *Comparison of magnitude and phase frequency responses.*

5 Conclusion

The mixed generalized Routh table and factor division method are proposed to interval polynomials for obtaining the stable reduced order system. The reduction of seventh order interval system to second order interval system gives excellent step as well as frequency responses. The proposed method is mathematically simple and gives all possible stable lower order models.

6 Appendix

I Kharitonov Theorem

An interval family of polynomials $D(s)$ is robustly stable if, and only if, the Kharitonov polynomials are stable.

$$
\begin{aligned}
D^{++}(s) &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_4^+ s^4 + a_5^+ s^5 + \cdots \\
D^{+-}(s) &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^+ s^4 + a_5^- s^5 + \cdots \\
D^{-+}(s) &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + a_4^- s^4 + a_5^+ s^5 + \cdots, \\
D^{--}(s) &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + a_4^- s^4 + a_5^- s^5 + \cdots
\end{aligned}
$$

For interval polynomial, the testing set is at most four Kharitonov polynomial.

II Theorem (Anderson, Jury and Mansour)

The testing set for an interval polynomial of invariant degree is

$$
\begin{array}{ll}
D^{+-}(s) & \text{for } n=3, \\
D^{+-}(s), D^{++}(s) & \text{for } n=4, \\
D^{+-}(s), D^{++}(s), D^{-+}(s), & \text{for } n=5, \\
D^{+-}(s), D^{++}(s), D^{-+}(s), D^{--}(s), & \text{for } n>5.
\end{array}
$$

For $n=1$ and $n=2$, a necessary and sufficient condition for robust stability is positive lower bounds on the coefficients.

The denominator polynomial of the higher order system in Section 4 is

$$
\begin{aligned}
D(s) = {} & [0.95, 1.05]s^7 + [8.779, 9.703]s^6 + [52.231, 57.729]s^5 + \\
& + [182.875, 202.125]s^4 + [429.02, 474.18]s^3 + \\
& + [572.47, 632.73]s^2 + [325.28, 359.52]s + [57.352, 63.389],
\end{aligned}
$$

$n=7$, therefore the necessary and sufficient condition is all the four polynomials should be stable.

$$D^{++}(s)=63.39+359.52s+572.47s^2+429.02s^3+$$
$$+202.13s^4+57.73s^5+8.78s^6+0.95s^7.$$

Constructing Routh table for $D^{++}(s)$,

s^7	0.95	57.73	429.02	359.52
s^6	8.78	202.13	572.47	63.39
s^5	35.86	367.07	352.66	
s^4	112.25	486.13	63.39	
s^3	211.79	332.41		
s^2	309.94	63.39		
s^1	289.10			
s^0	63.39			

$$D^{+-}(s)=63.39+325.28s+572.47s^2+474.18s^3+$$
$$+202.13s^4+52.23s^5+8.78s^6+1.05s^7.$$

Constructing Routh table for $D^{+-}(s)$,

s^7	1.05	52.23	474.18	325.28
s^6	8.78	202.13	572.47	63.39
s^5	28.06	405.71	317.70	
s^4	75.17	473.06	63.39	
s^3	229.16	294.04		
s^2	376.60	63.39		
s^1	255.47			
s^0	63.39			

$$D^{-+}(s)=57.35+359.52s+632.73s^2+429.02s^3+$$
$$+182.87s^4+57.73s^5+9.70s^6+0.95s^7.$$

Constructing Routh table for $D^{-+}(s)$,

s^7	0.95	57.73	429.02	359.52
s^6	9.70	182.88	632.73	57.35
s^5	39.82	367.07	353.90	
s^4	93.44	546.50	57.35	
s^3	134.15	329.46		
s^2	317.02	57.35		
s^1	305.19			
s^0	57.35			

$$D^{--}(s) = 57.35 + 325.28s + 632.73s^2 + 474.18s^3 + \\ + 182.87s^4 + 52.23s^5 + 9.70s^6 + 1.05s^7 .$$

Constructing Routh table for $D^{--}(s)$,

s^7	1.05	52.23	474.18	325.28
s^6	9.70	182.88	632.73	57.35
s^5	32.44	405.71	319.07	
s^4	61.53	537.30	57.35	
s^3	122.42	288.84		
s^2	392.13	57.35		
s^1	270.93			
s^0	57.35			

From the above Routh tables it is clear that all the four Kharitonov polynomials are stable. Therefore, from Kharitonov's Stability Criterion $D(s)$ is stable.

The denominator polynomial of the reduced lower order system in Section 4 is

$$D_2(s) = [364.72, 366.62]s^2 + [281.08, 282.35]s + [59.74, 61] ,$$

$n = 2$, therefore a necessary and sufficient condition for robust stability is positive lower bounds on the coefficients.

$$D_2(s) = 364.72s^2 + 281.08s + 59.74 .$$

Constructing Routh table for $D_2(s)$,

$$\begin{array}{c|ll} s^2 & 364.72 & 281.08 \\ s^1 & 59.74 & \\ s^0 & 281.08 & \end{array}$$

From the above Routh tables it is clear that $D_2(s)$ is stable. Thus the proposed method guarantees the robust stability of reduced order systems.

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